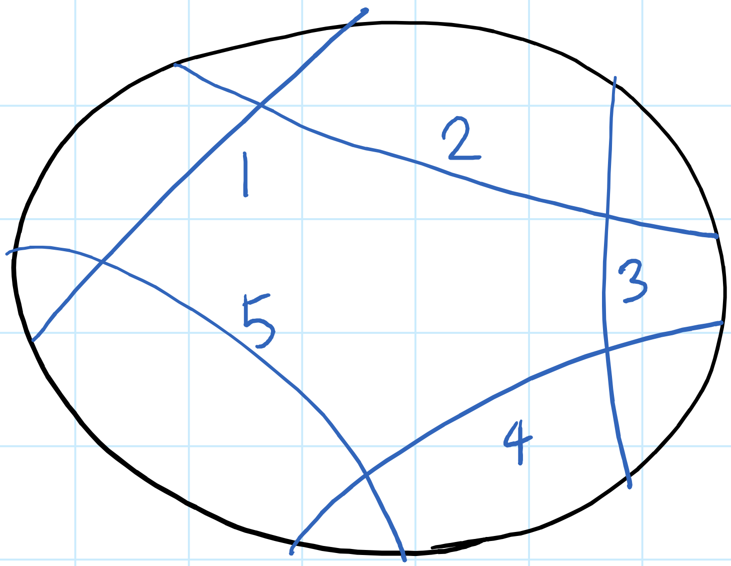
Recognizing Circle Graphs

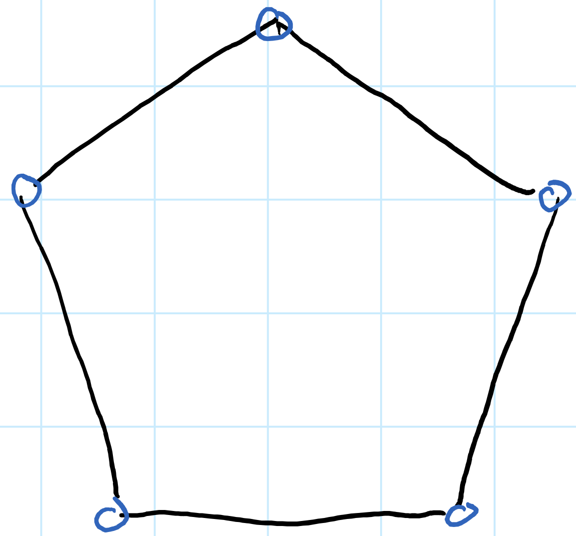
Jim Geelen Edward Lee

University of Waterloo



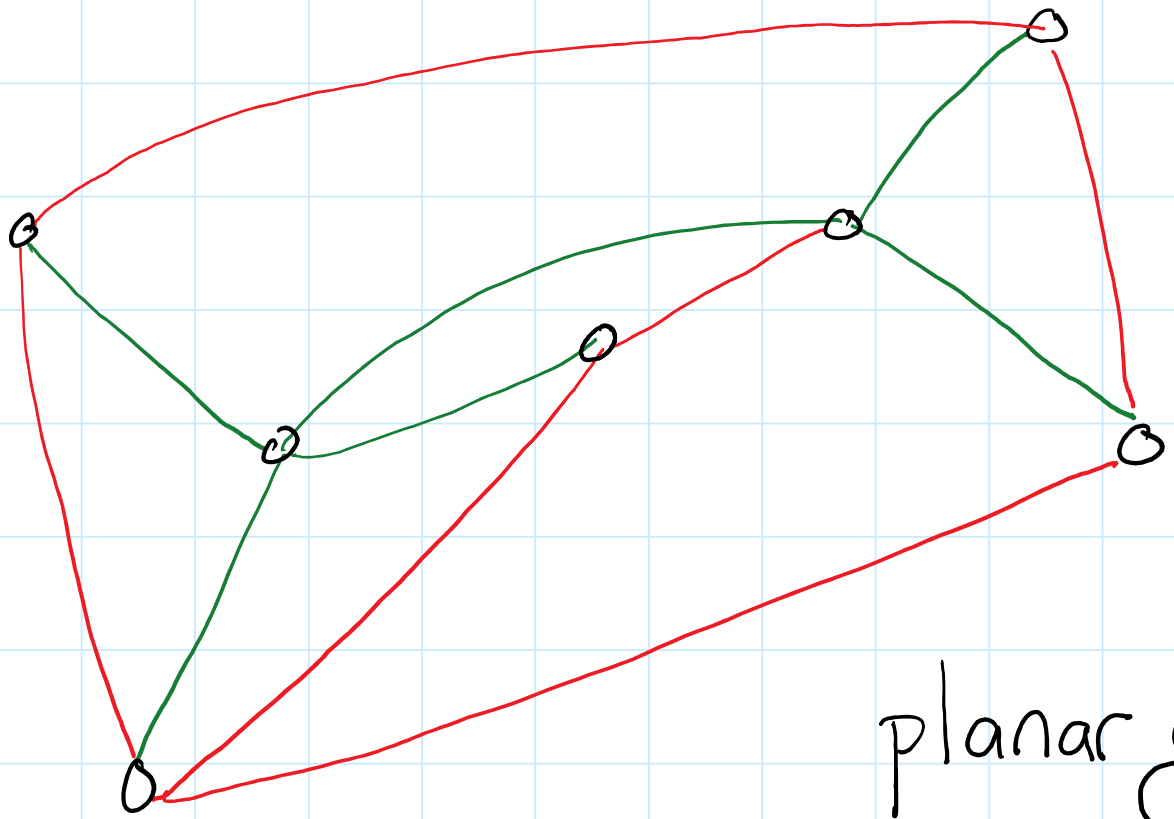
Chord Diagram

$\mathcal{C}$

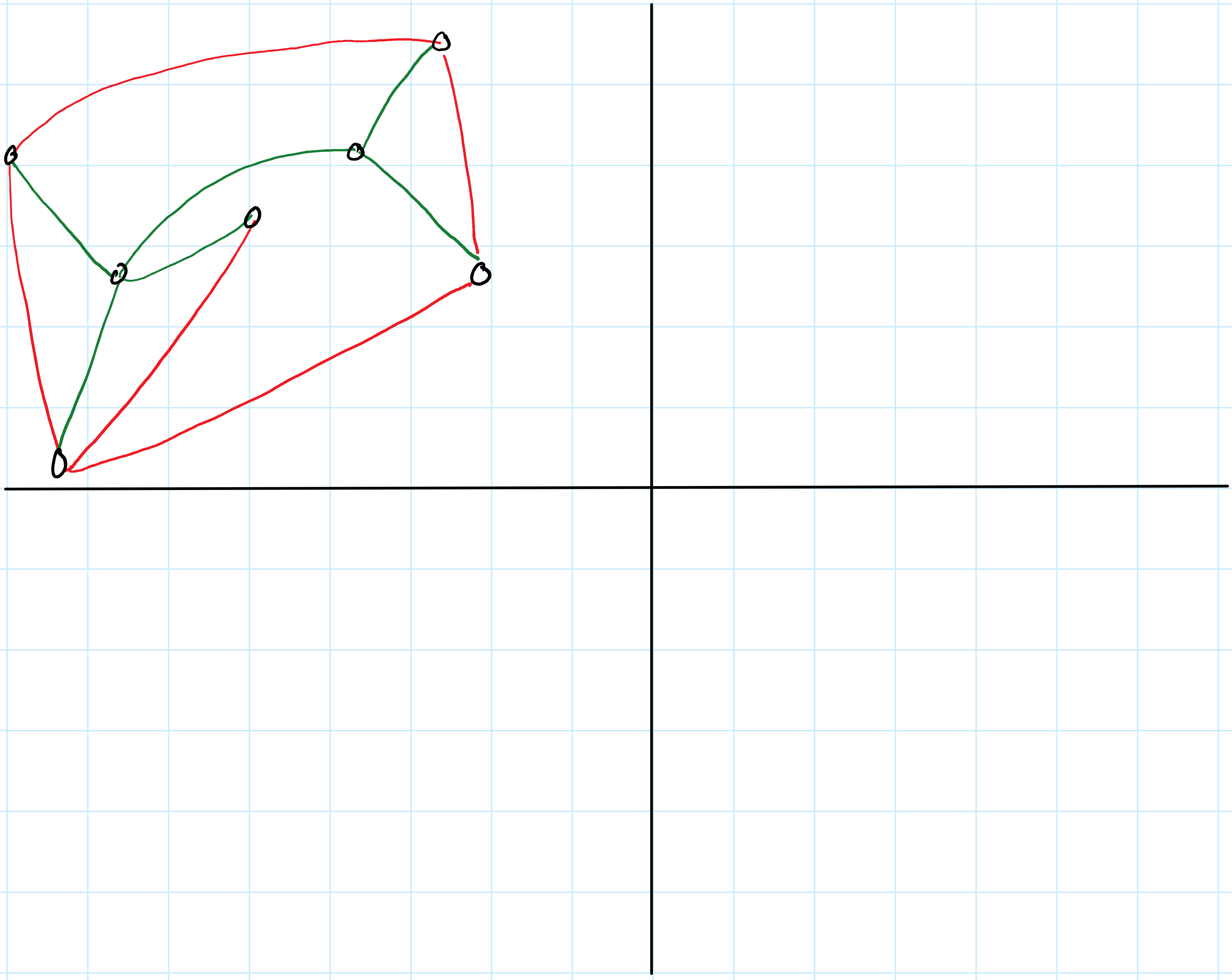


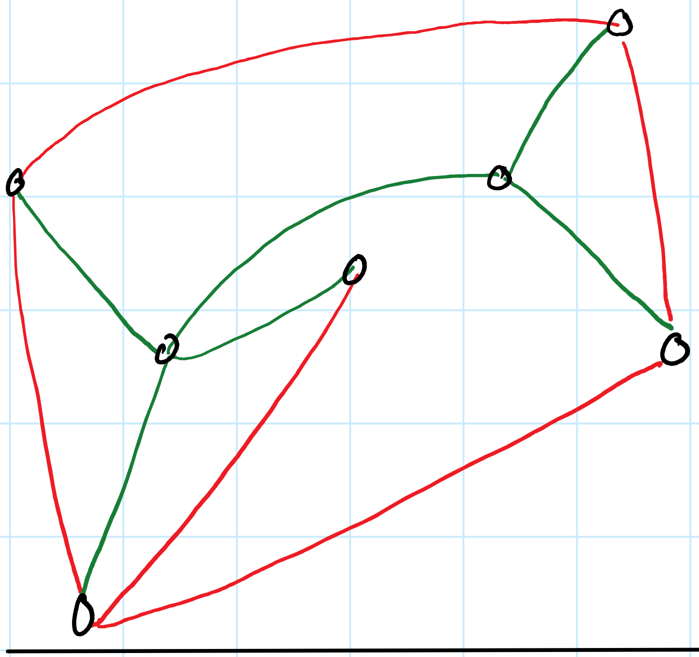
Circle Graph

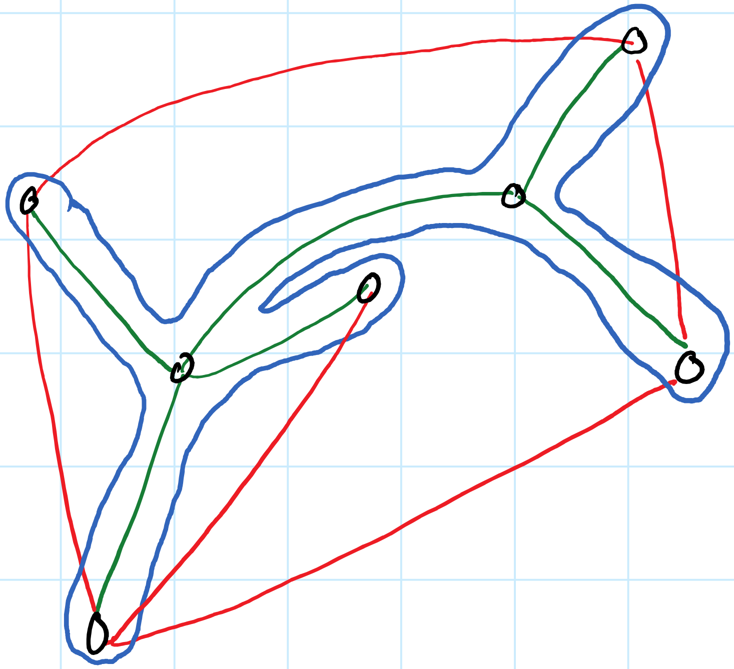
G

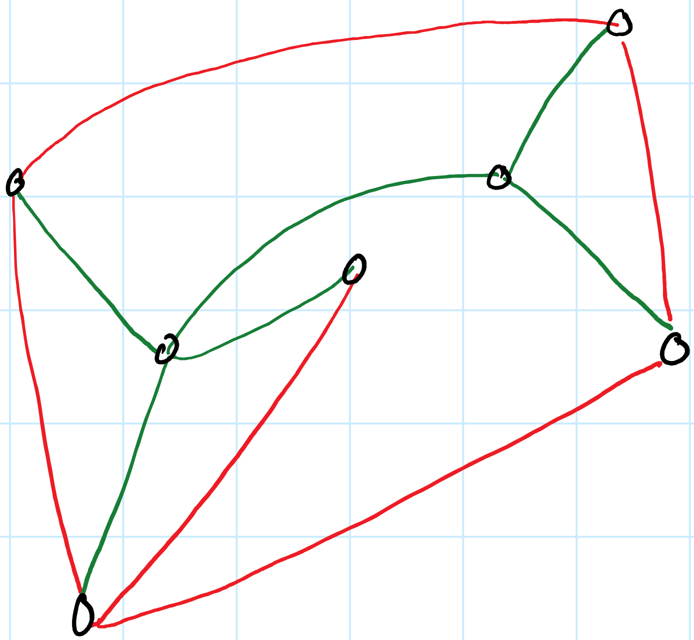


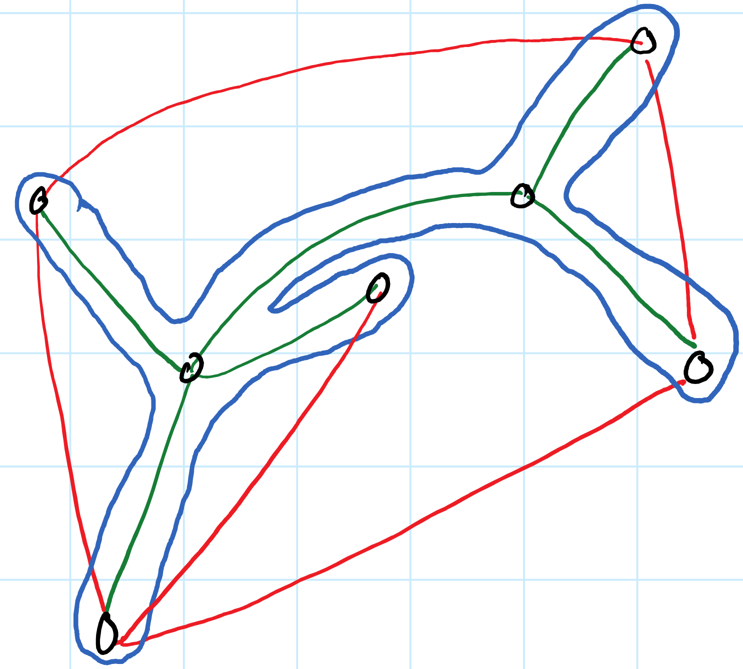
planar graph
green - spanning tree
red - everything else

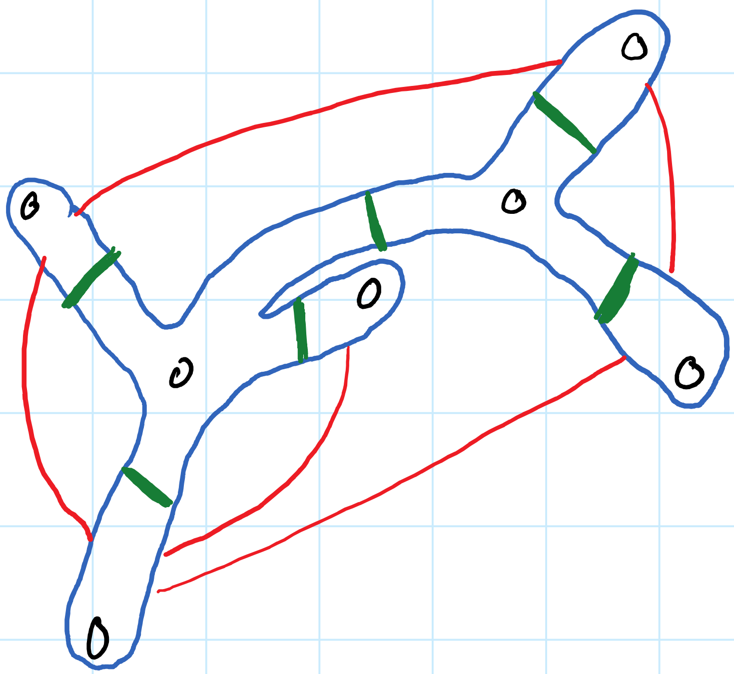


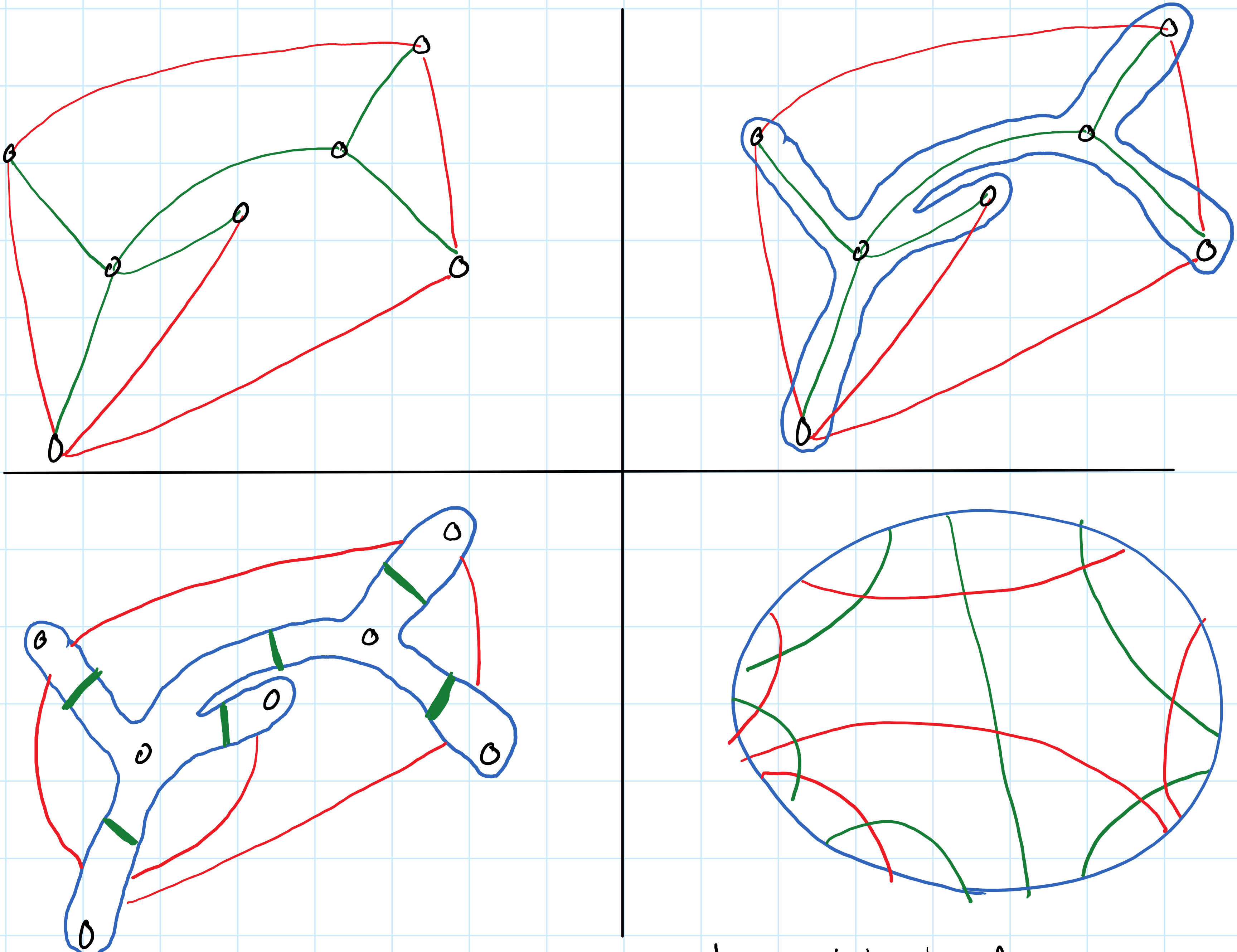




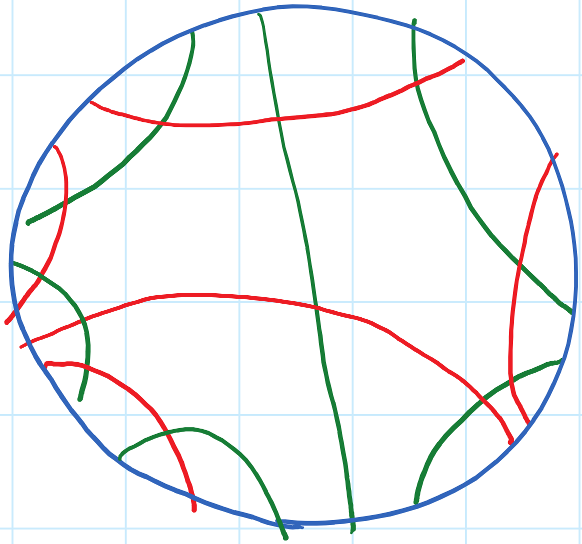






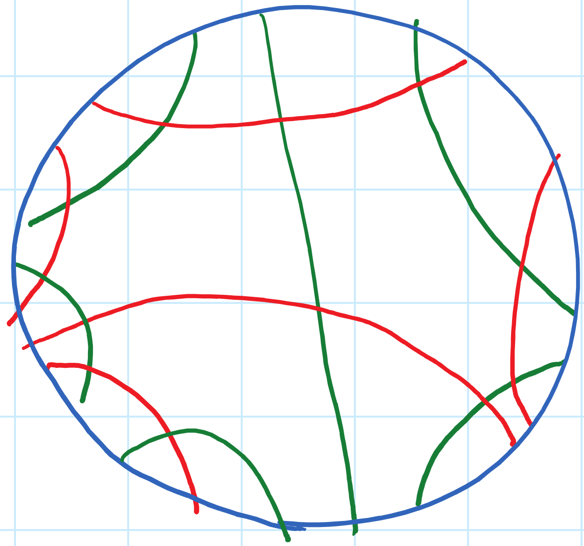


bipartite chord diagram

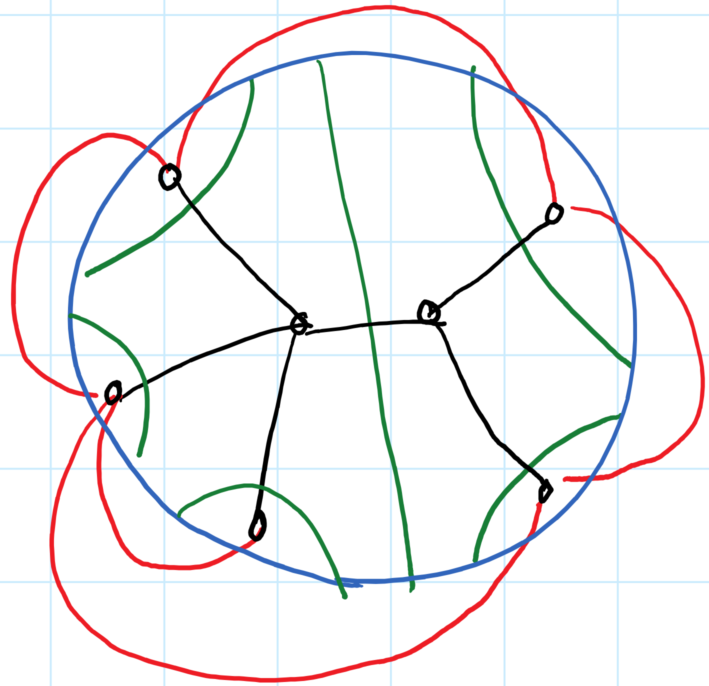


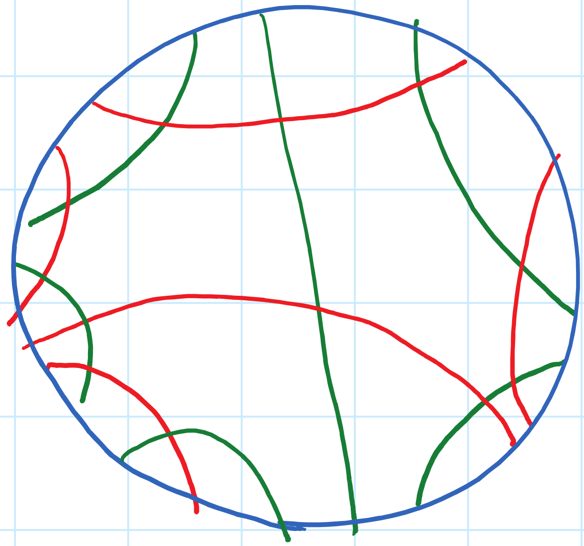
bipartite chord diagram

This process is also reversible.

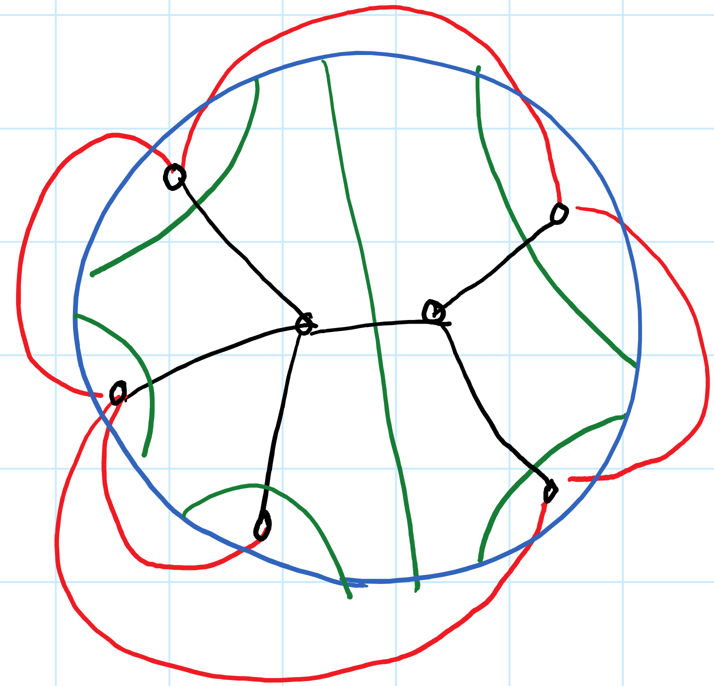


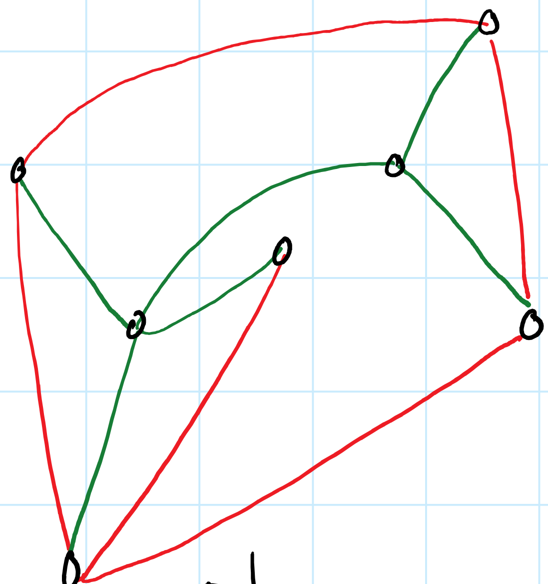
bipartite chord diagram





bipartite chord diagram





planar graph

This construction is known as

De Frayessix's Theorem

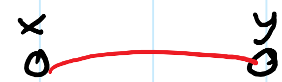
The important takeaway is that:

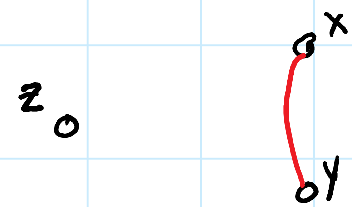
characterizations of circle graphs

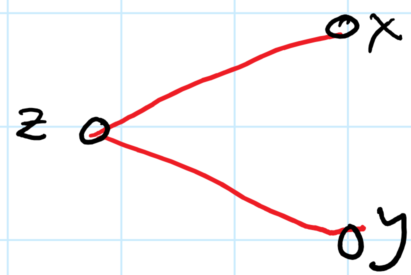
$\Downarrow$

characterizations of planar graphs

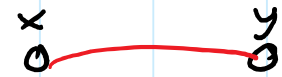
Naji System for G:

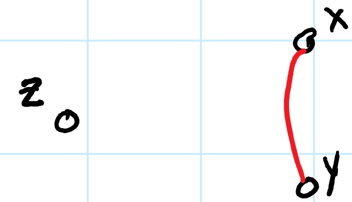
(1) $\beta(x,y) + \beta(y,x) = 1$ for 

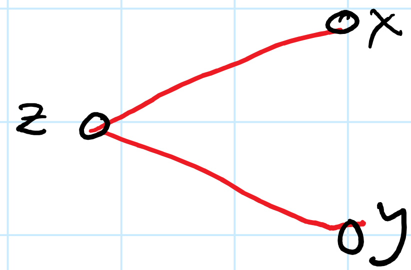
(2) $\beta(z,x) + \beta(z,y) = 0$ for 

(3) $\beta(z,x) + \beta(z,y) + \beta(x,y) + \beta(y,x) = 1$ for 

Naji System for G:

(1) $\beta(x,y)+\beta(y,x)=1$ for 

(2) $\beta(z,x)+\beta(z,y)=0$ for 

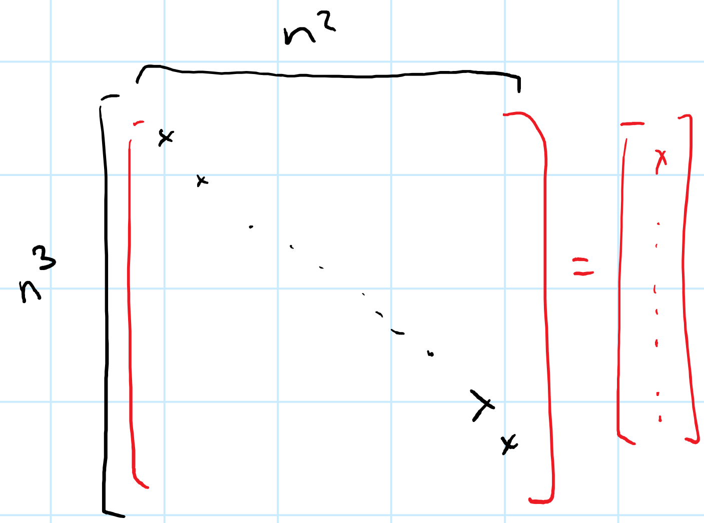
(3) $\beta(z,x)+\beta(z,y)+\beta(x,y)+\beta(y,x)=1$ for 

(N)

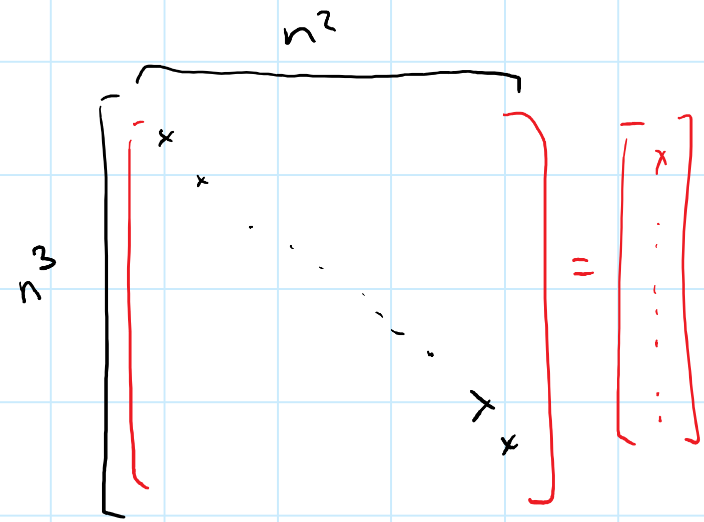
Theorem (Naji) G is a circle graph $\iff$ (N) has a solution over $GF(2)$.

An interlude into complexity:

- (N) has n^2 variables
  $\sim n^3$ equations



An interlude into complexity:



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- Gaussian (column) elimination can be done in $O(n^5)$ time for each column.

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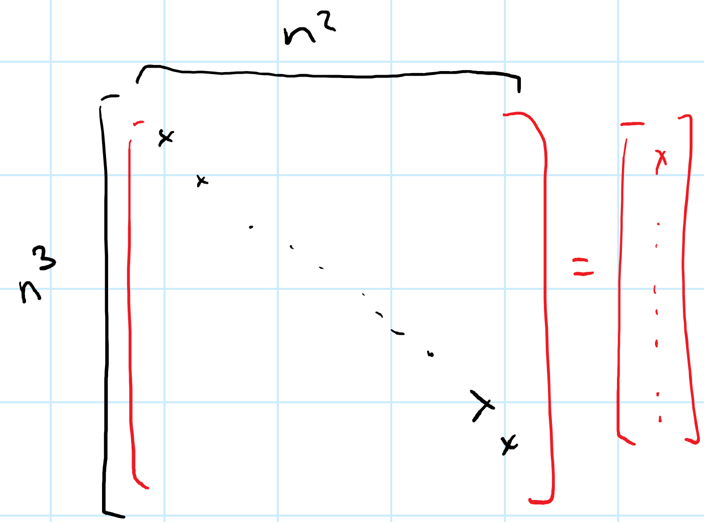
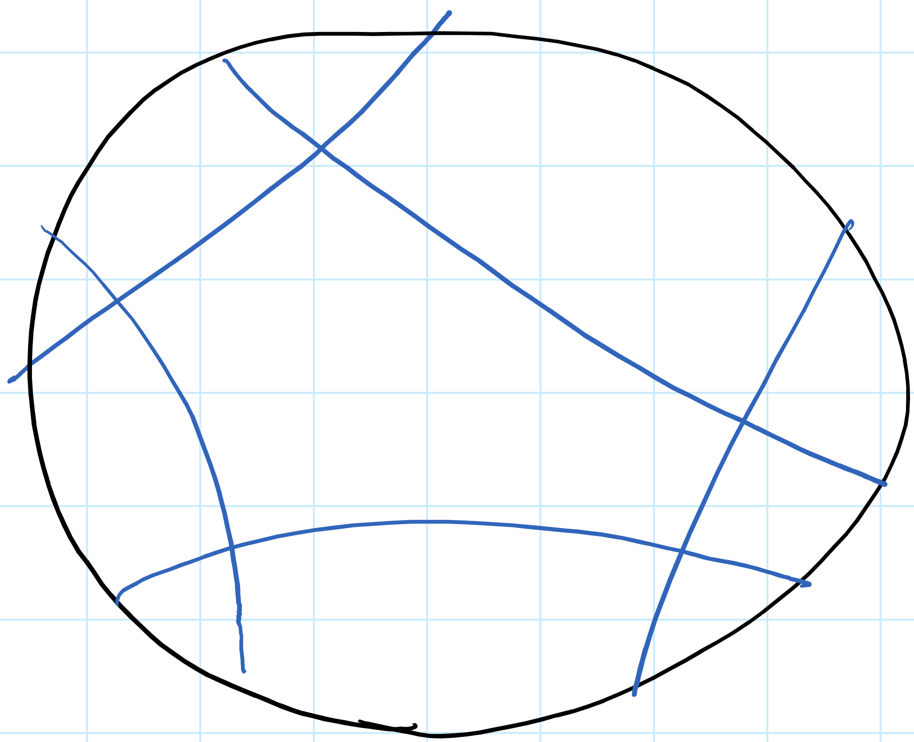


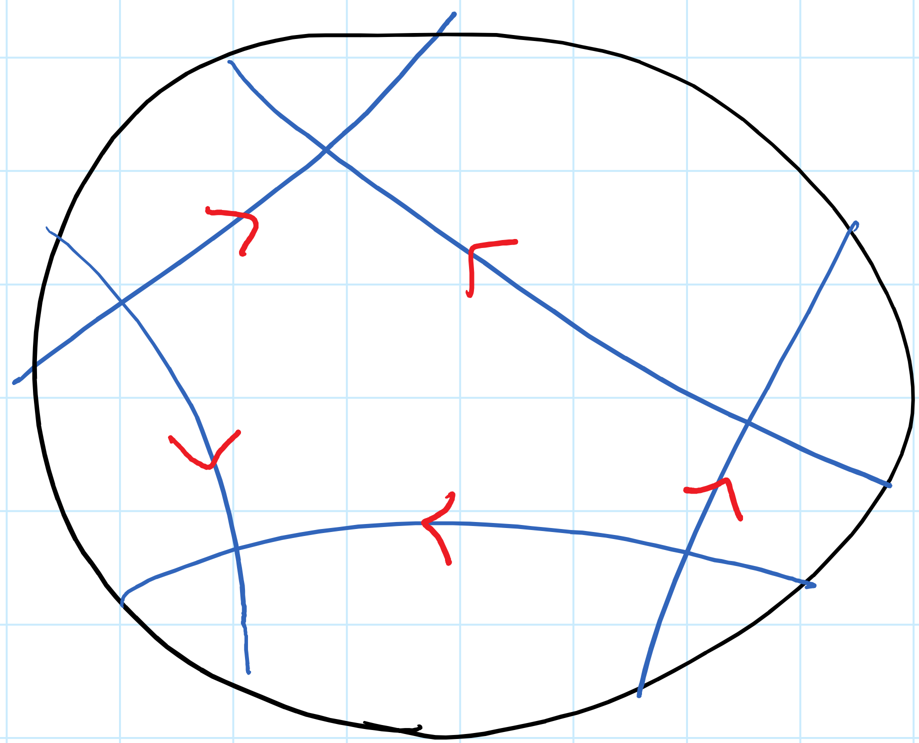
- Gaussian (column) elimination can be done in $O(n^5)$ time for each column.

- So $O(n^7)$ algorithm for recognizing circle graphs.

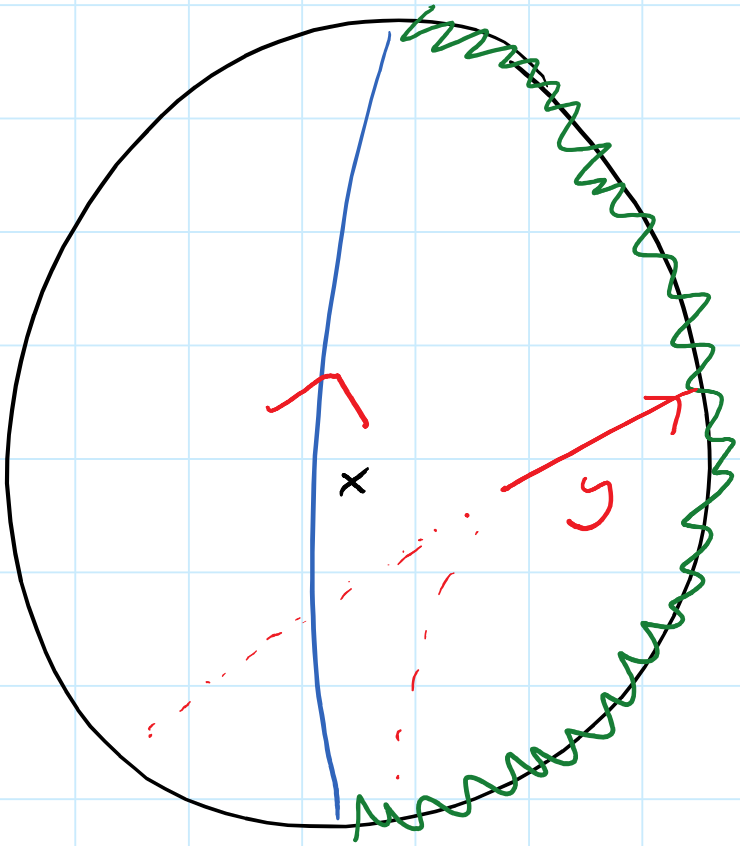
Circle Graph ⟹ Solution



chord diagram



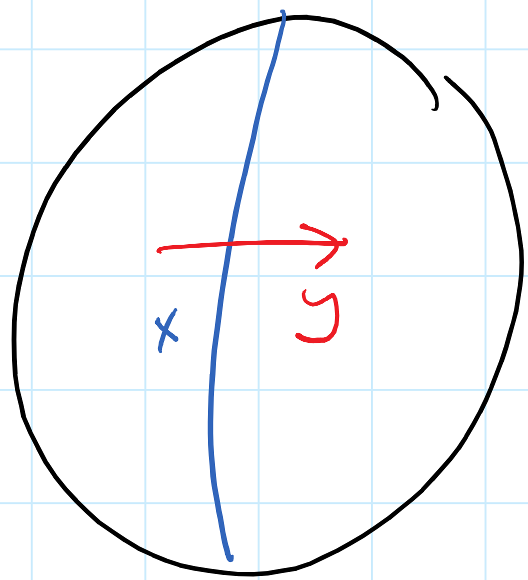
+orientations



Take $B(x,y)=0$ if the head of y is to the right of x.

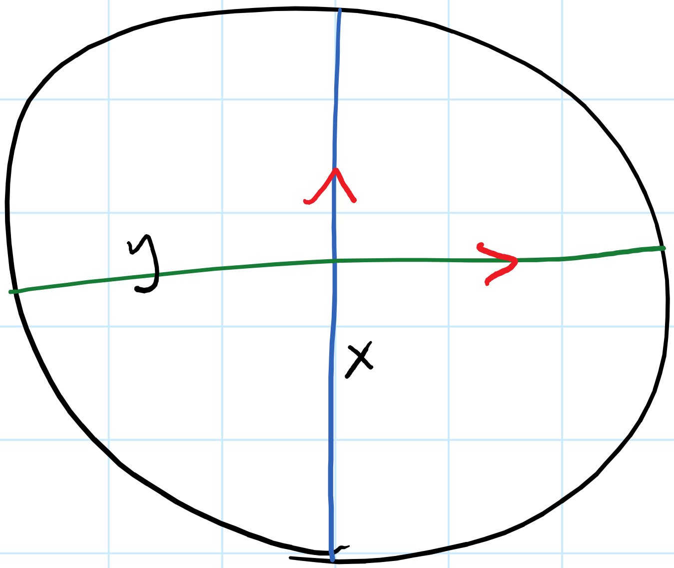
Notation: $B(\mathcal{C})$ for such solutions.

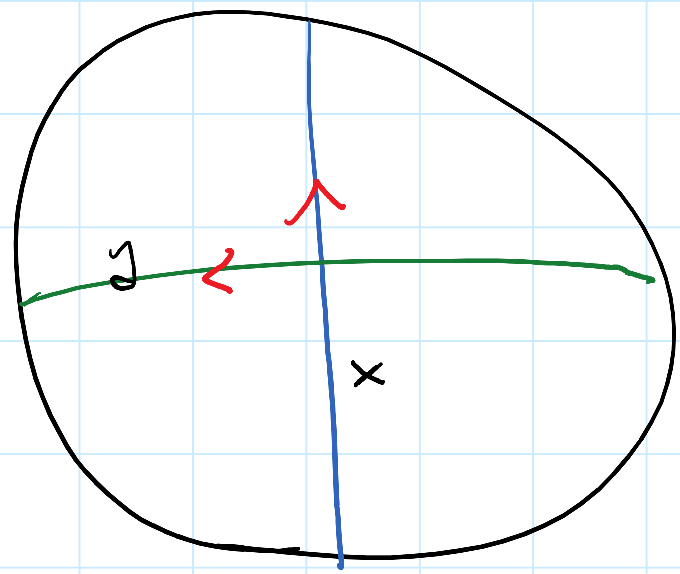
Aside: will often draw as:



Equation (1): $x \quad y$

$$B(x,y)+B(y,x)=1$$



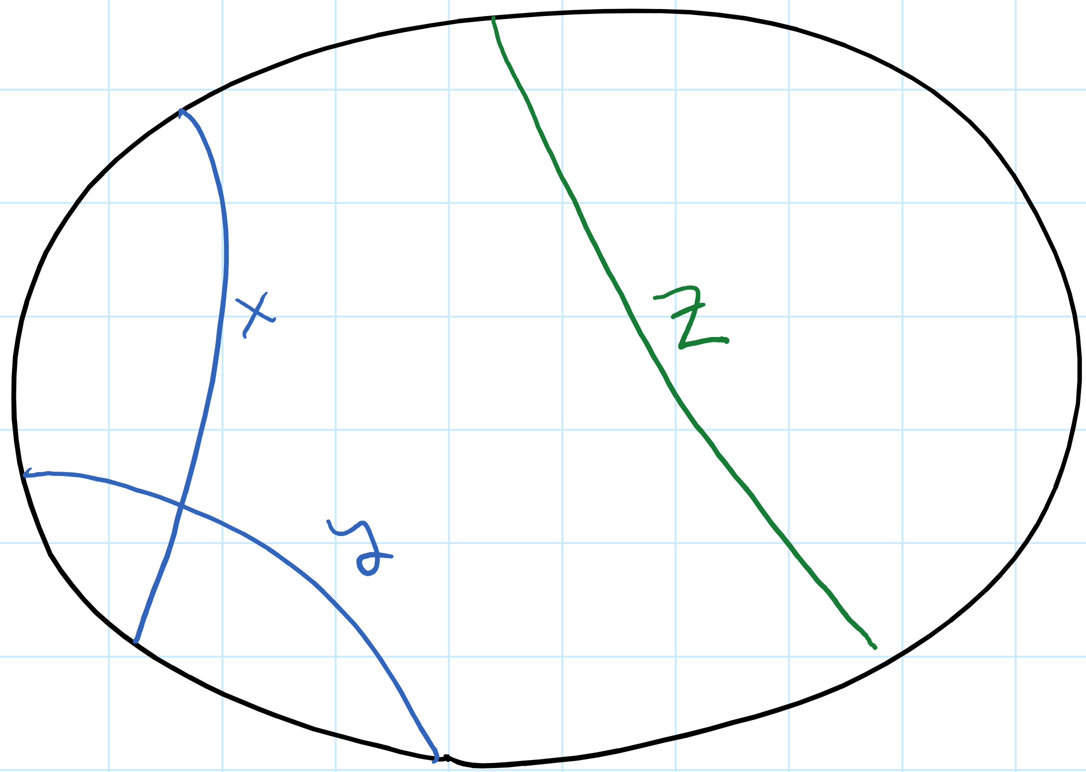


$$B(x,y)=1$$
$$B(y,x)=0$$

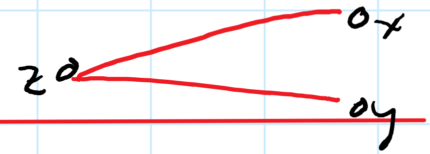
$$B(y,x)=1$$
$$B(x,y)=0$$

Equation (2): z ∘ $\quad$ ∘x ∘y

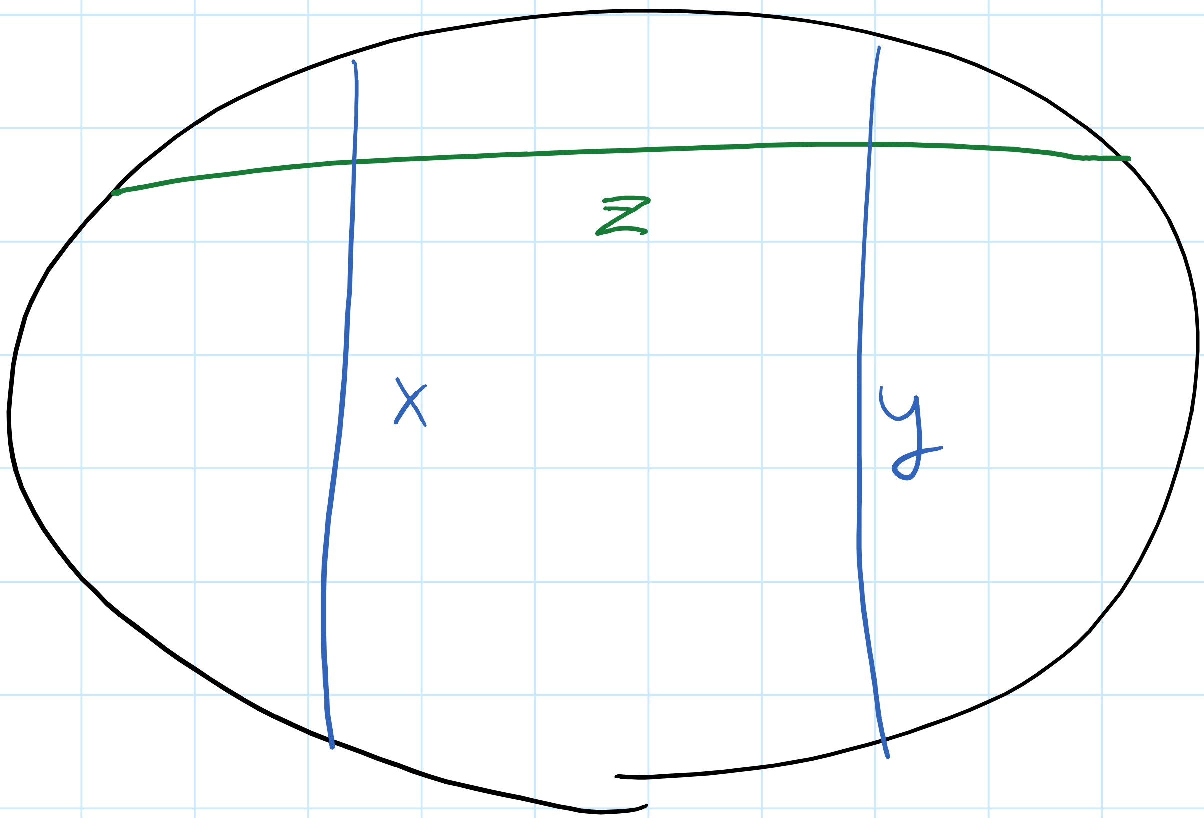
$$\beta(z, x) + \beta(z, y) = 0$$



Equation (3):



$$\beta(z,x)+\beta(z,y)+\beta(x,y)+\beta(y,x)=1$$

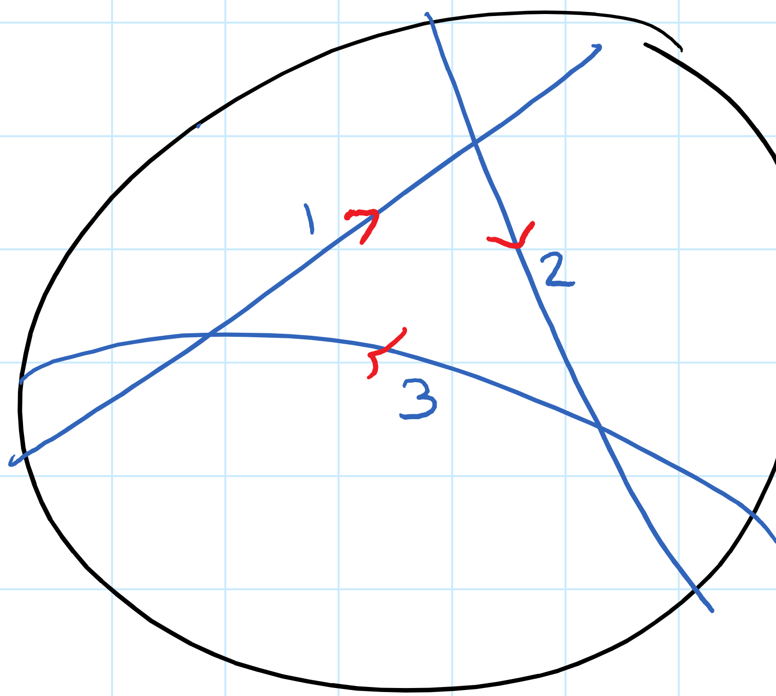


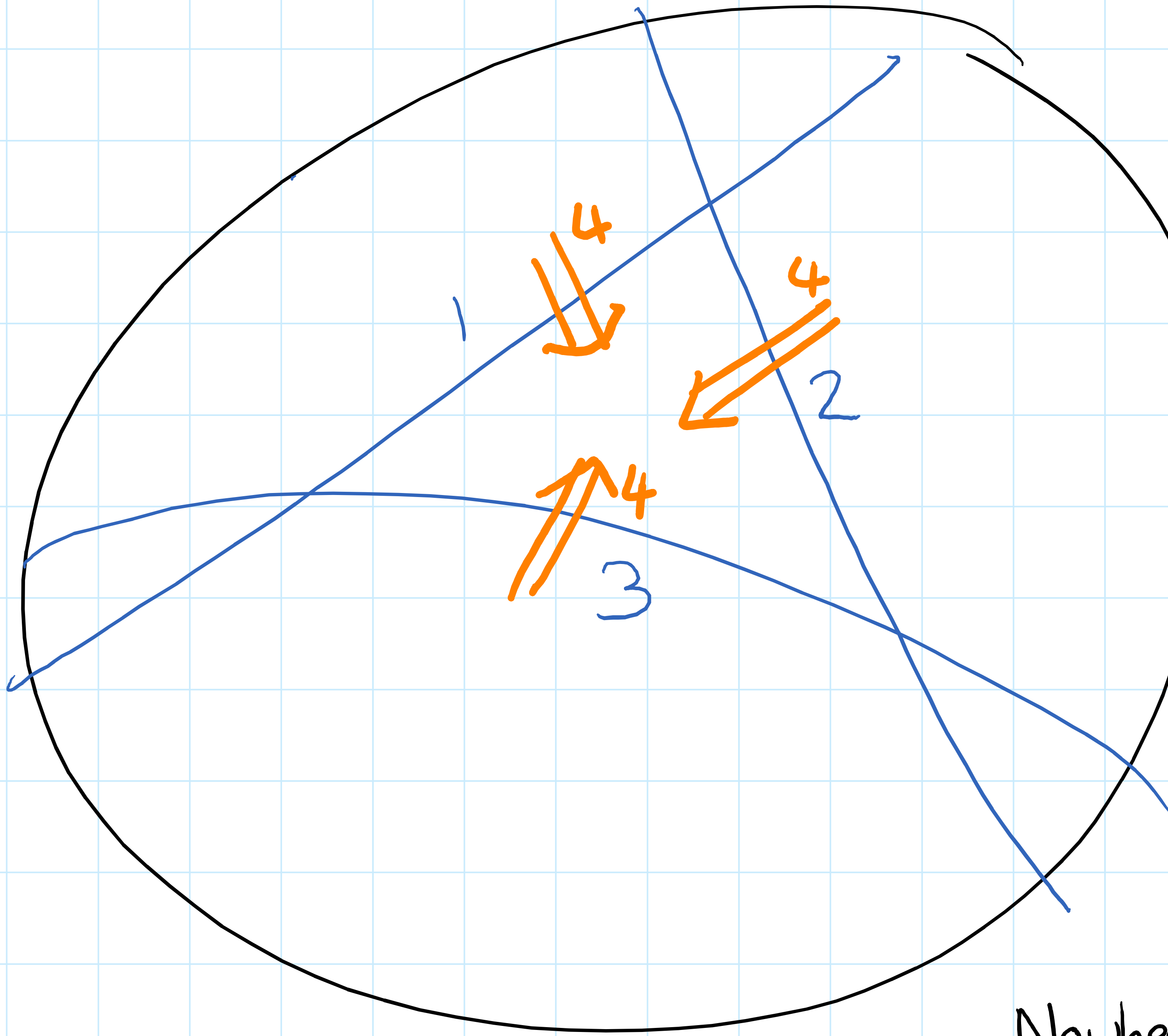
Not all Naji solutions are so natural.

Consider the following solution to K_4:

$\beta(1,2) = \beta(2,3) = \beta(3,1) = 0$ | everything else 1.

$\beta(1,4) = \beta(2,4) = \beta(3,4) = 0$ |





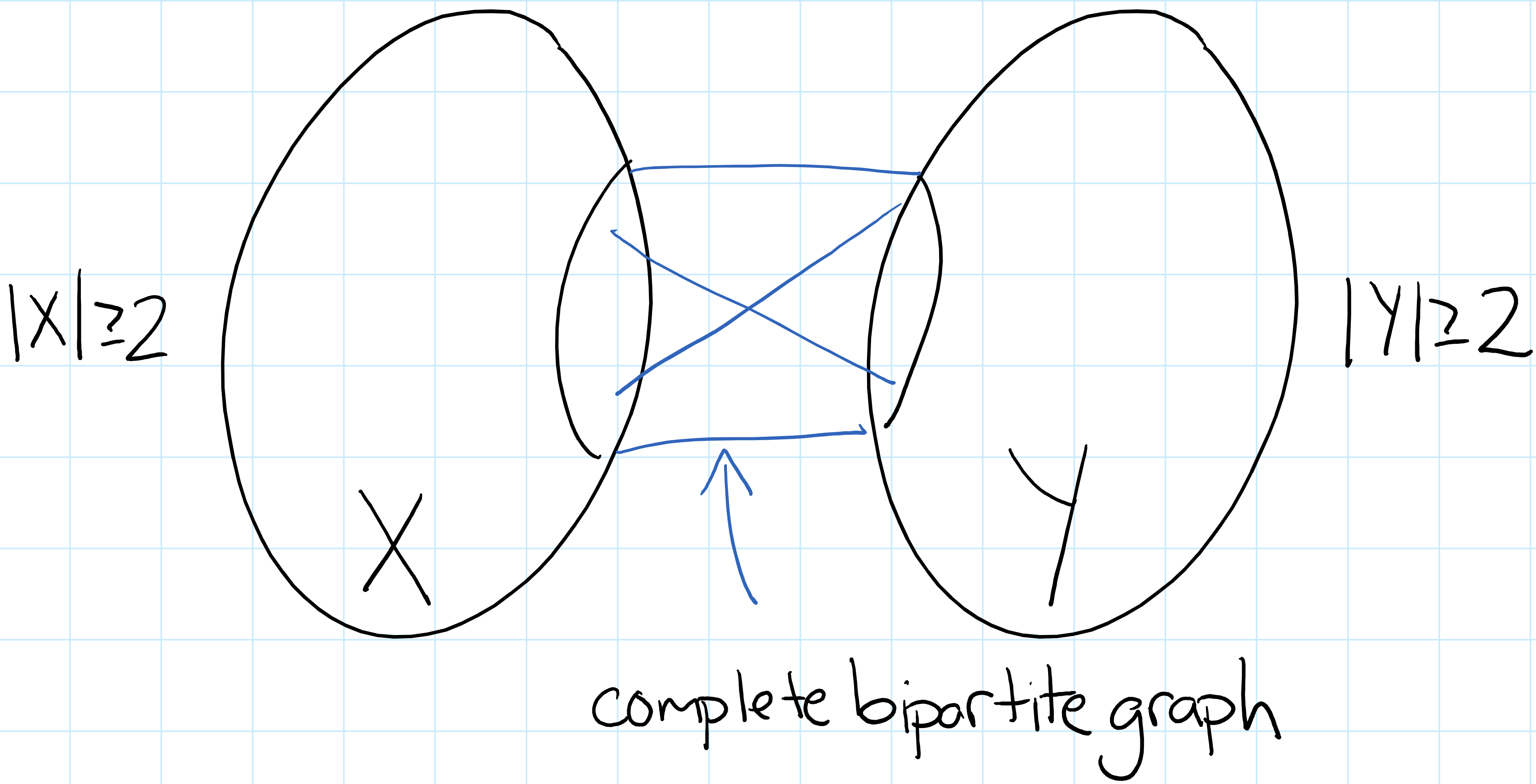
Nowhere to put 4.

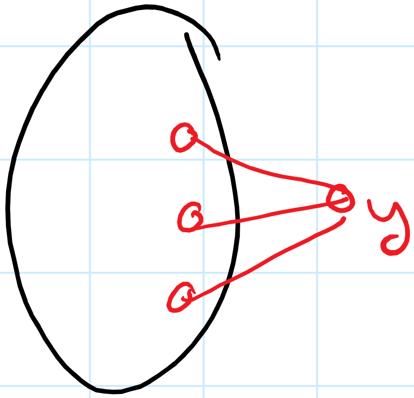
$$A\bar{x} = A\bar{y} = A\bar{z} = b$$

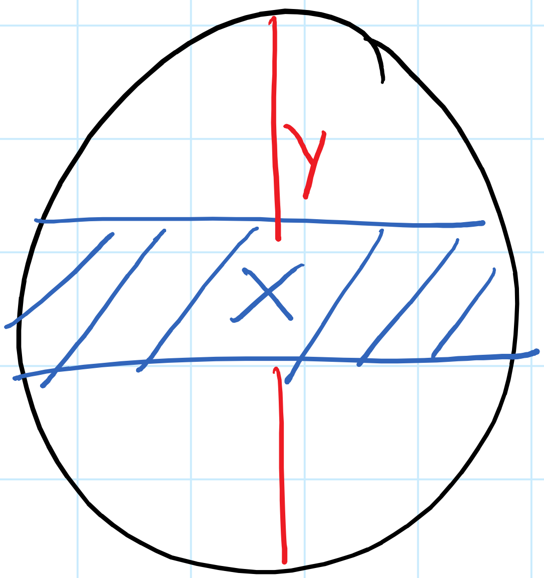
$$\Longrightarrow A(\bar{x} + \bar{y} + \bar{z}) = b$$

Can combine "unrelated" Naji solutions!

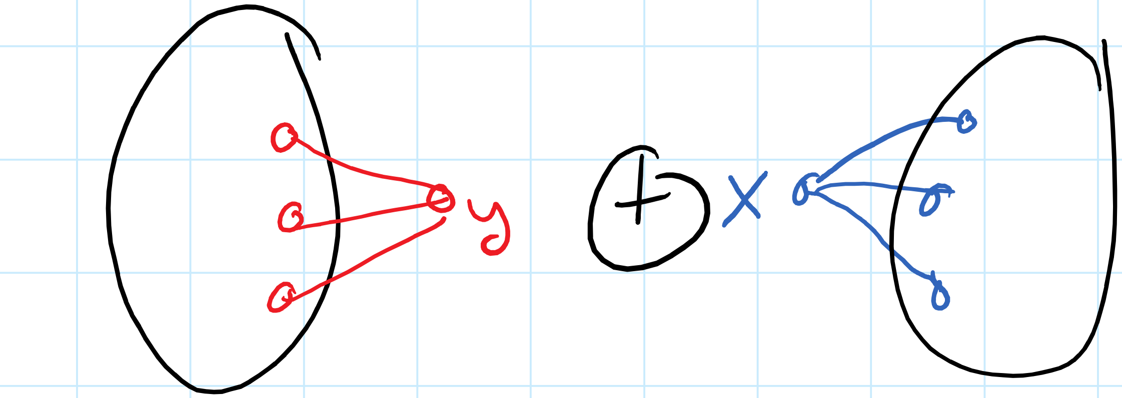
Splits ⇒ Inequivalent Diagrams

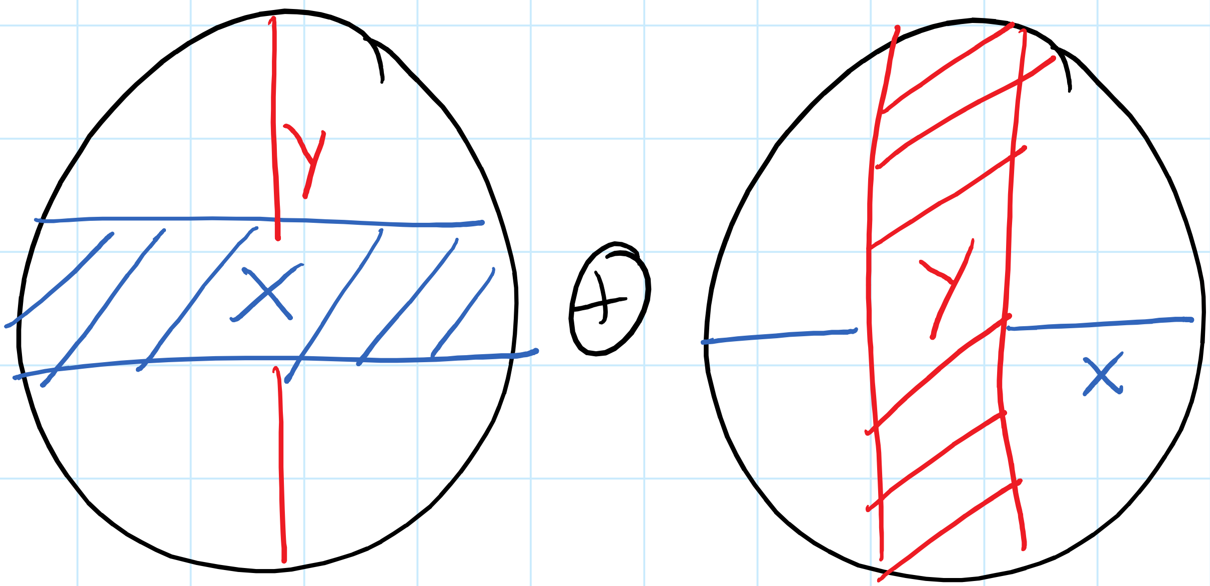




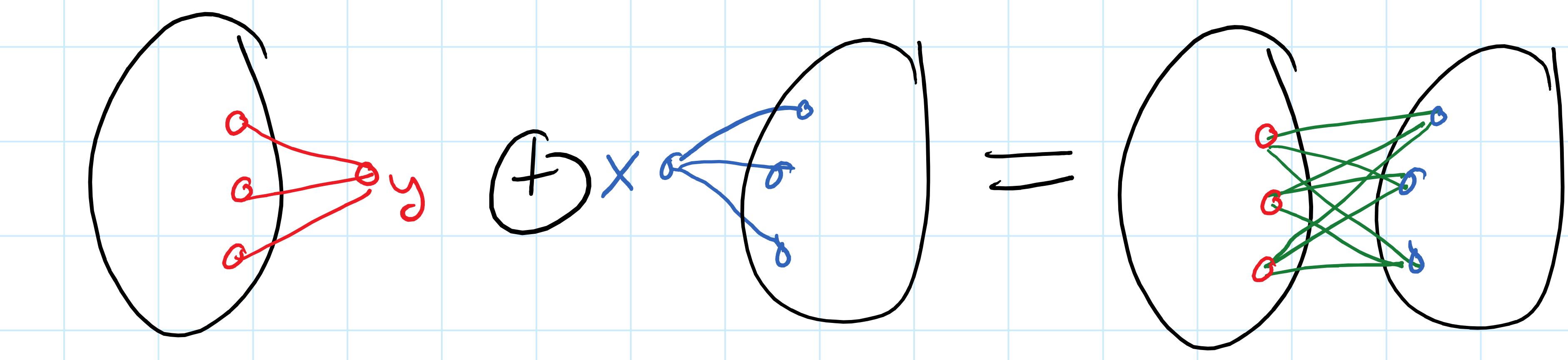


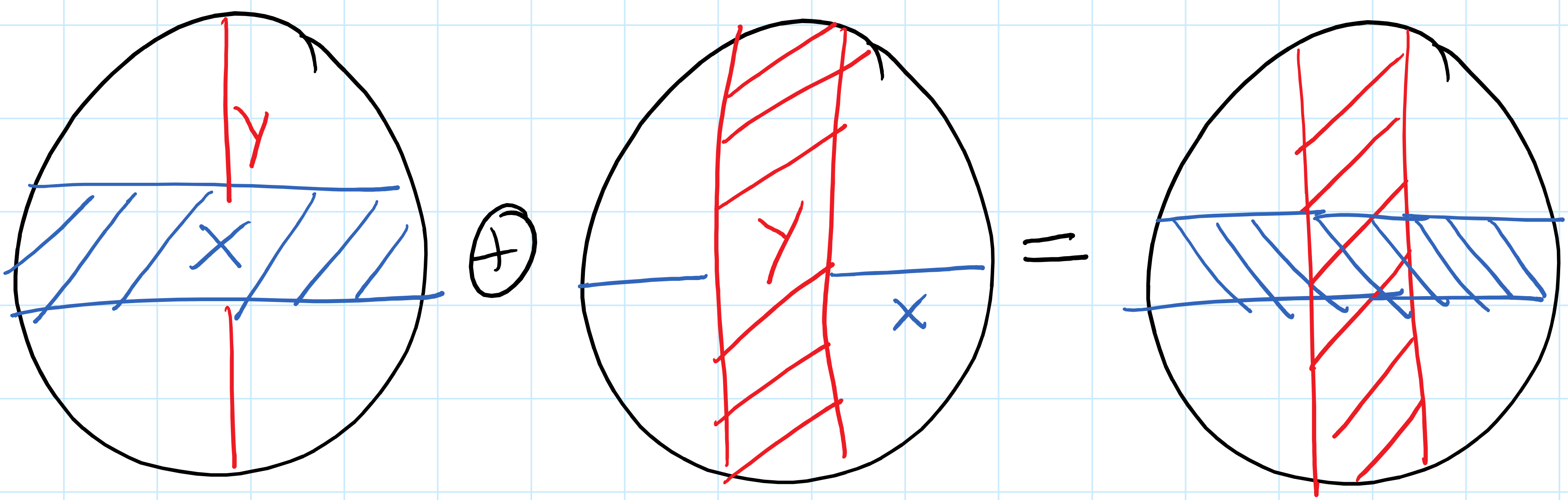
$G \circ Y$





$G \circ Y$ $\qquad$ $G \circ X$





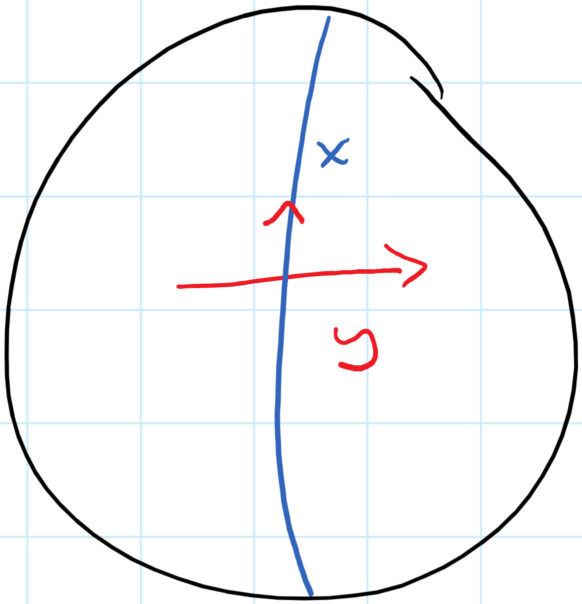
$G \circ Y$ $\qquad$ $G \circ X$ $\qquad$ G

Theorem (G,L)

Let β be a Naji solution for G. Then either:

(1): $\beta = \beta(e)$

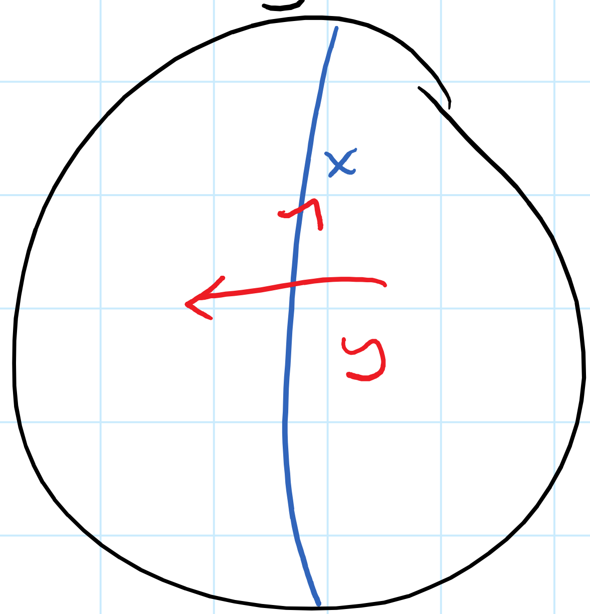
(2): G has a split.



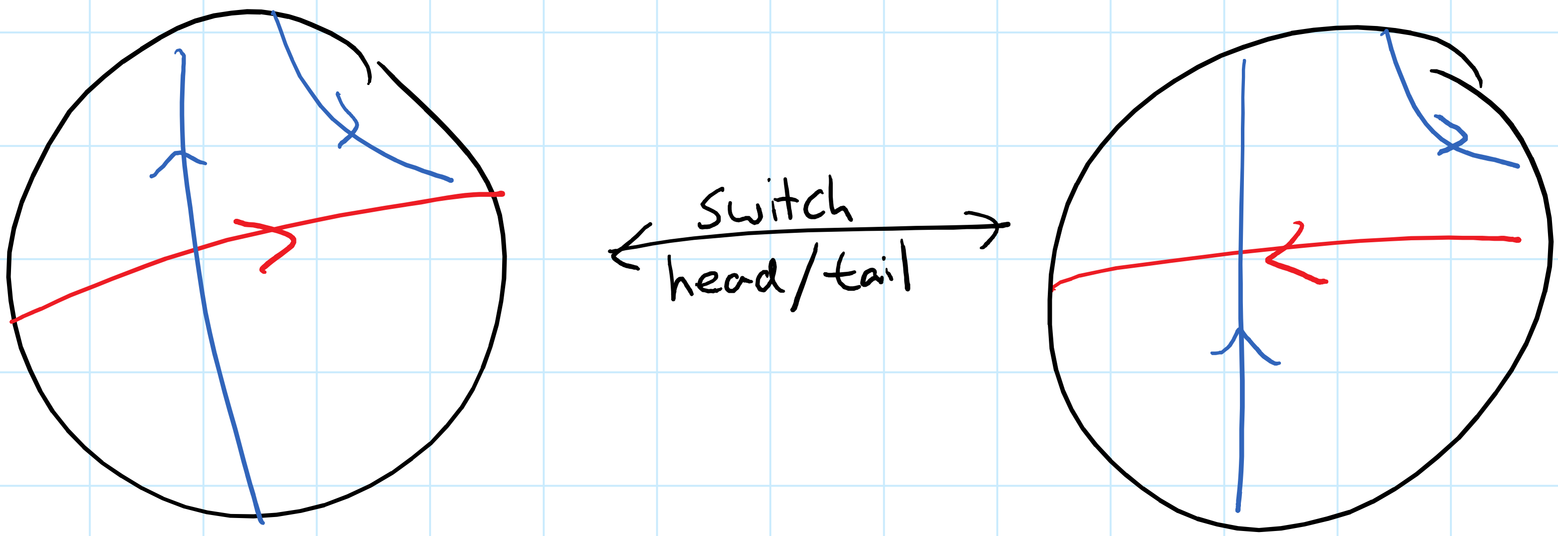
Each chord knows where the head of the next chord should go.

$B(x,y) = 0$

$B(x,y) = 1$



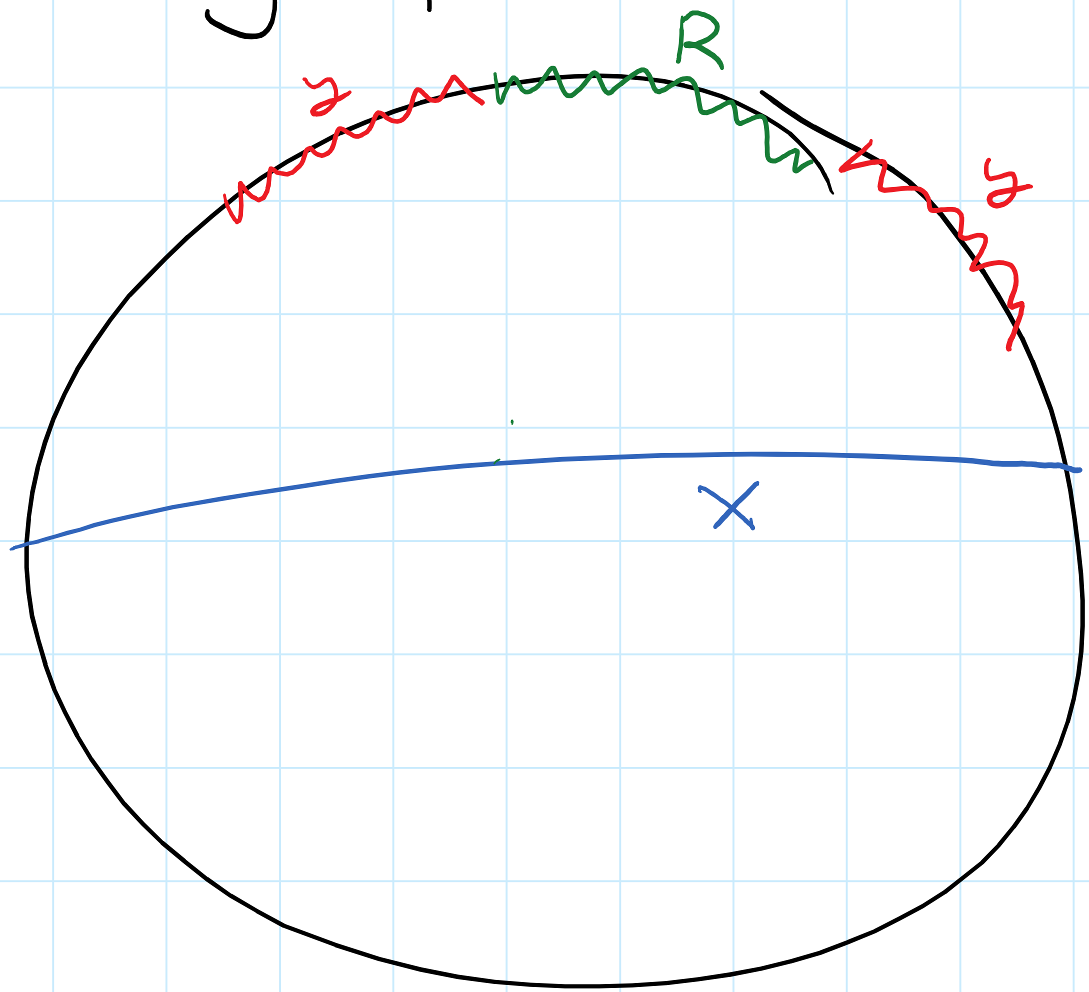
Moreover we can easily modify solution to find out where the tail should go. (Gasse)



$$\beta(e) = \delta(v) + \beta(e')$$

where $\delta_v(x,y) = 1$ if and only if $x = v$
or $y = v$ and $\{x,y\} \in E$.

So long as we place the chords in a connected manner there is at most one way to place a new chord.



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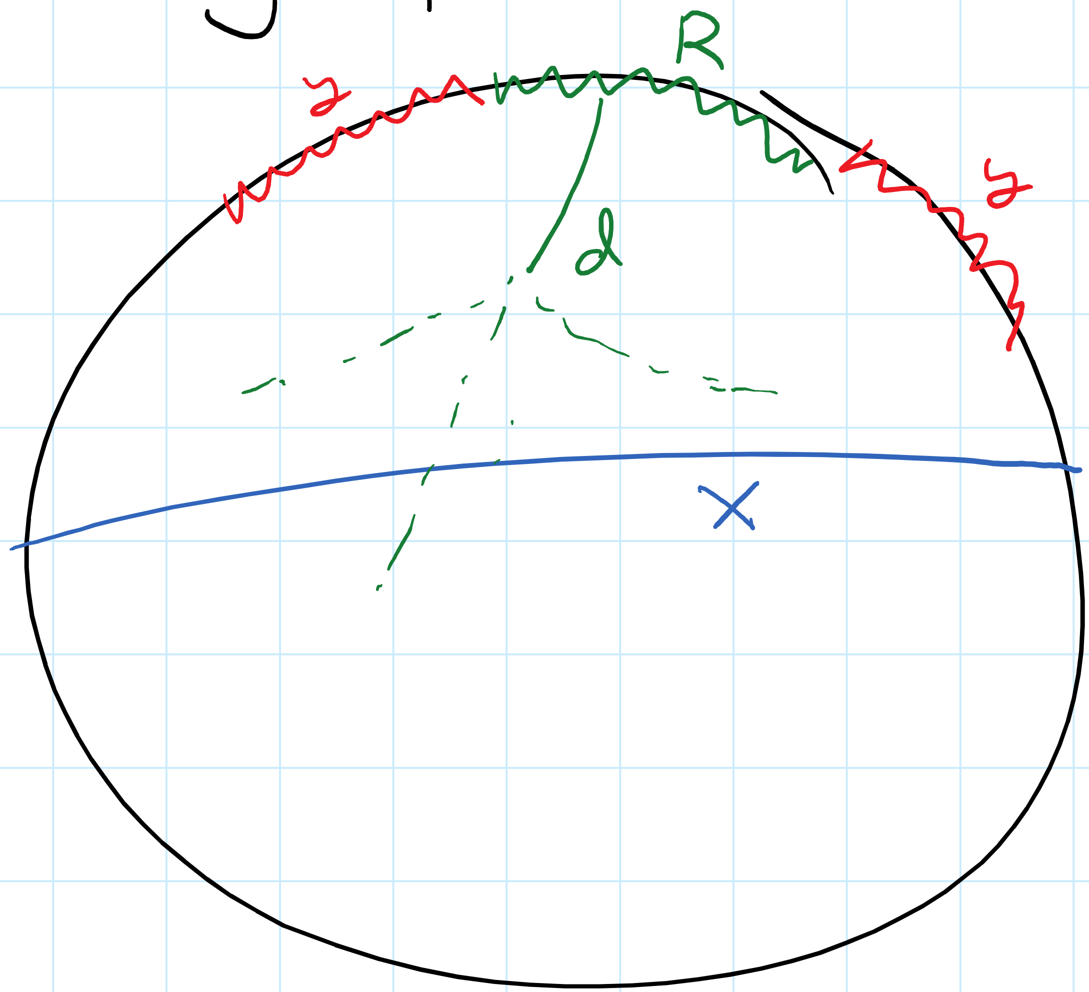

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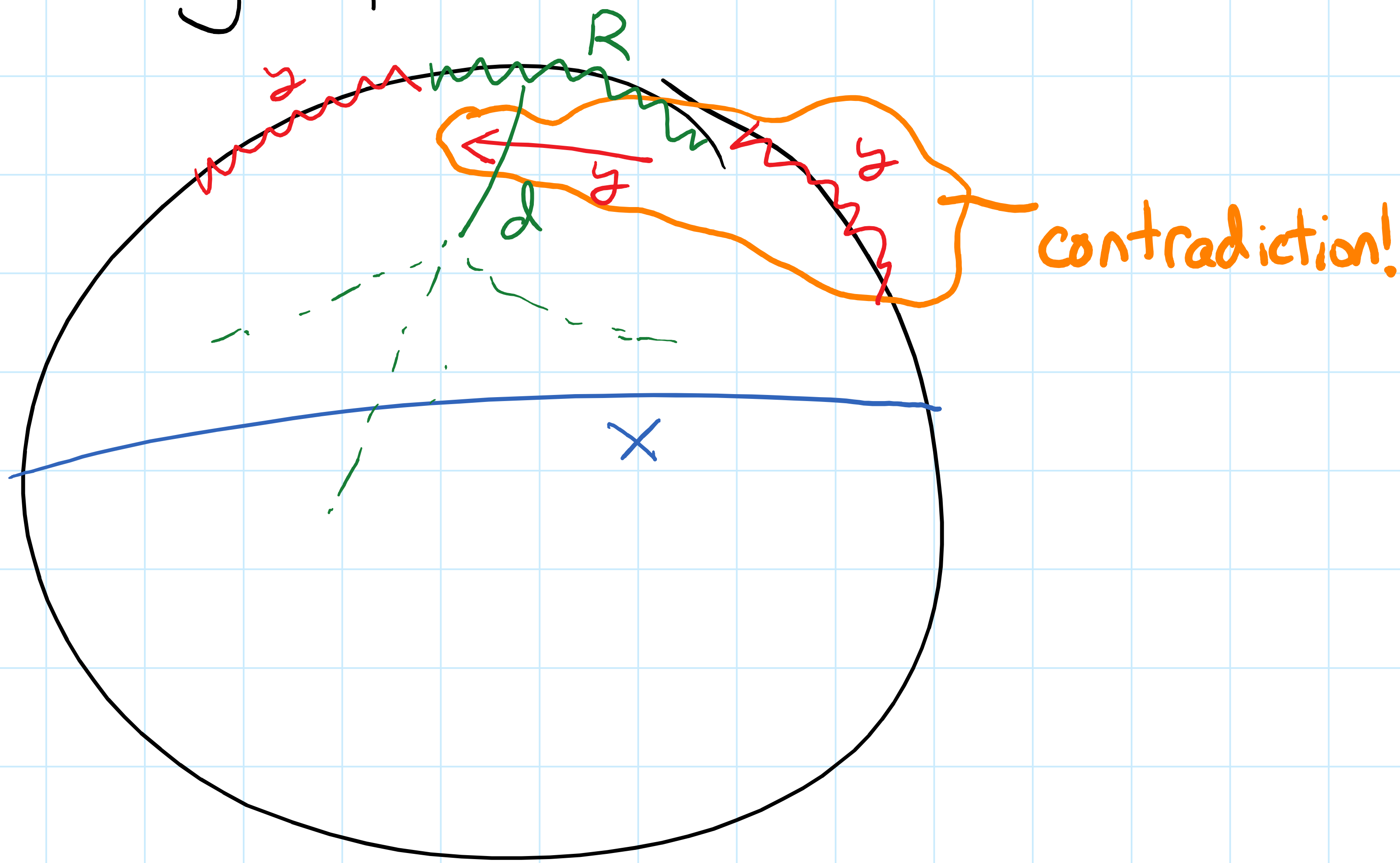
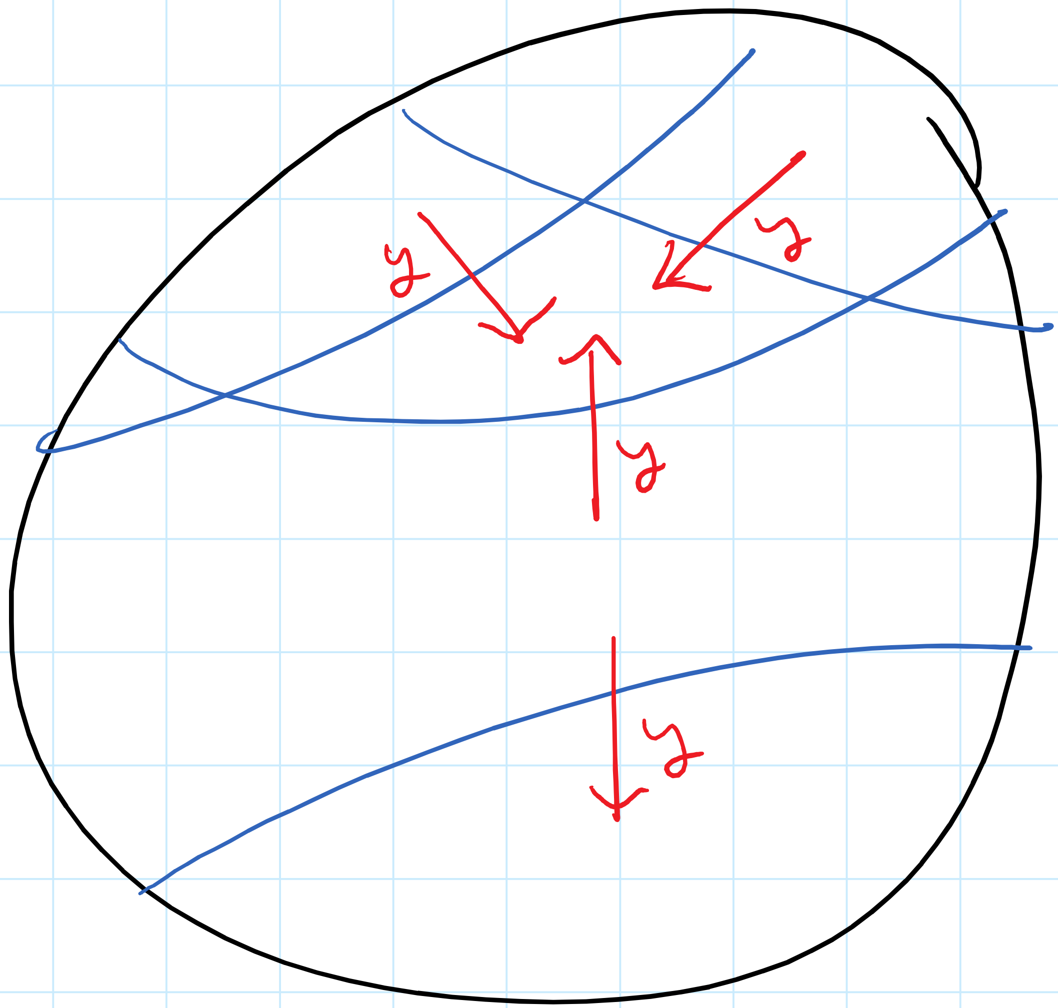


If something goes wrong:



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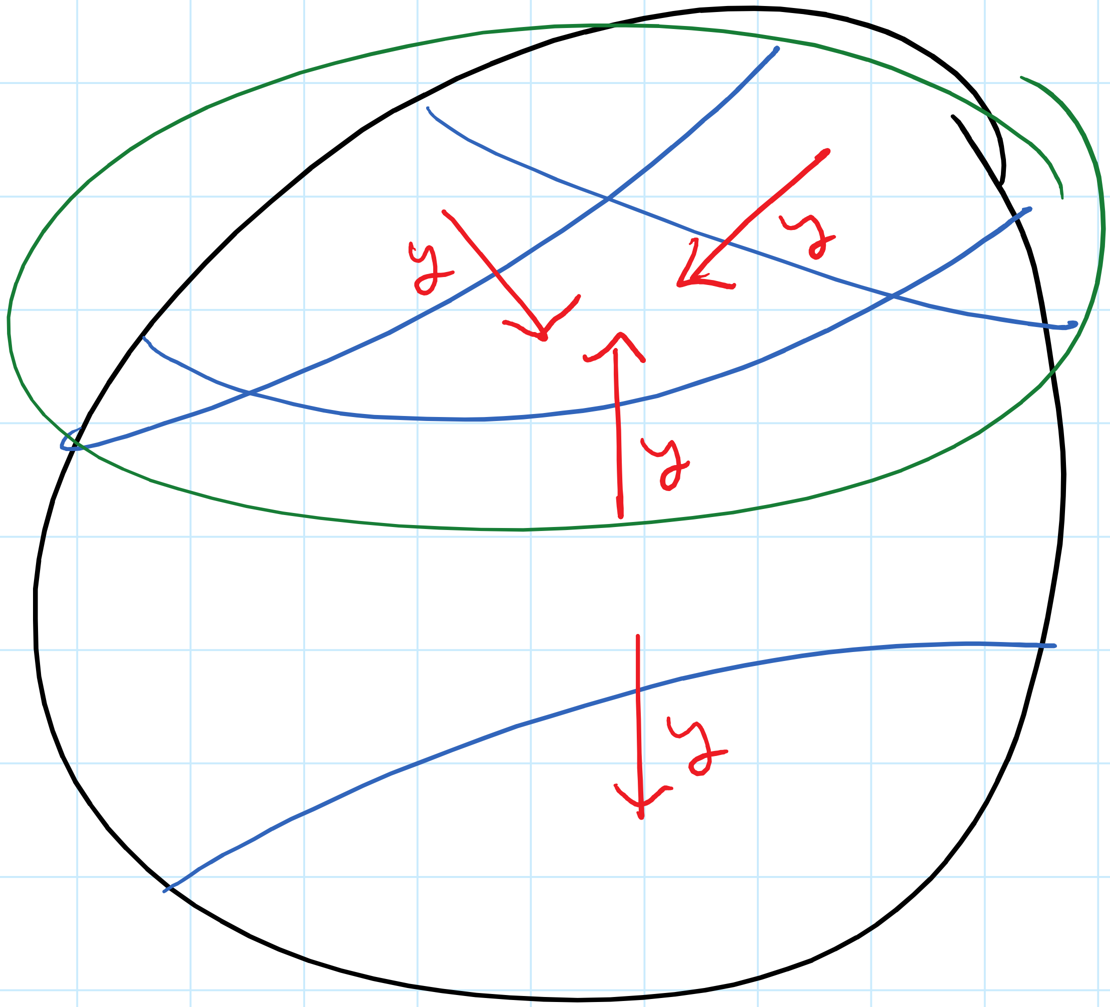

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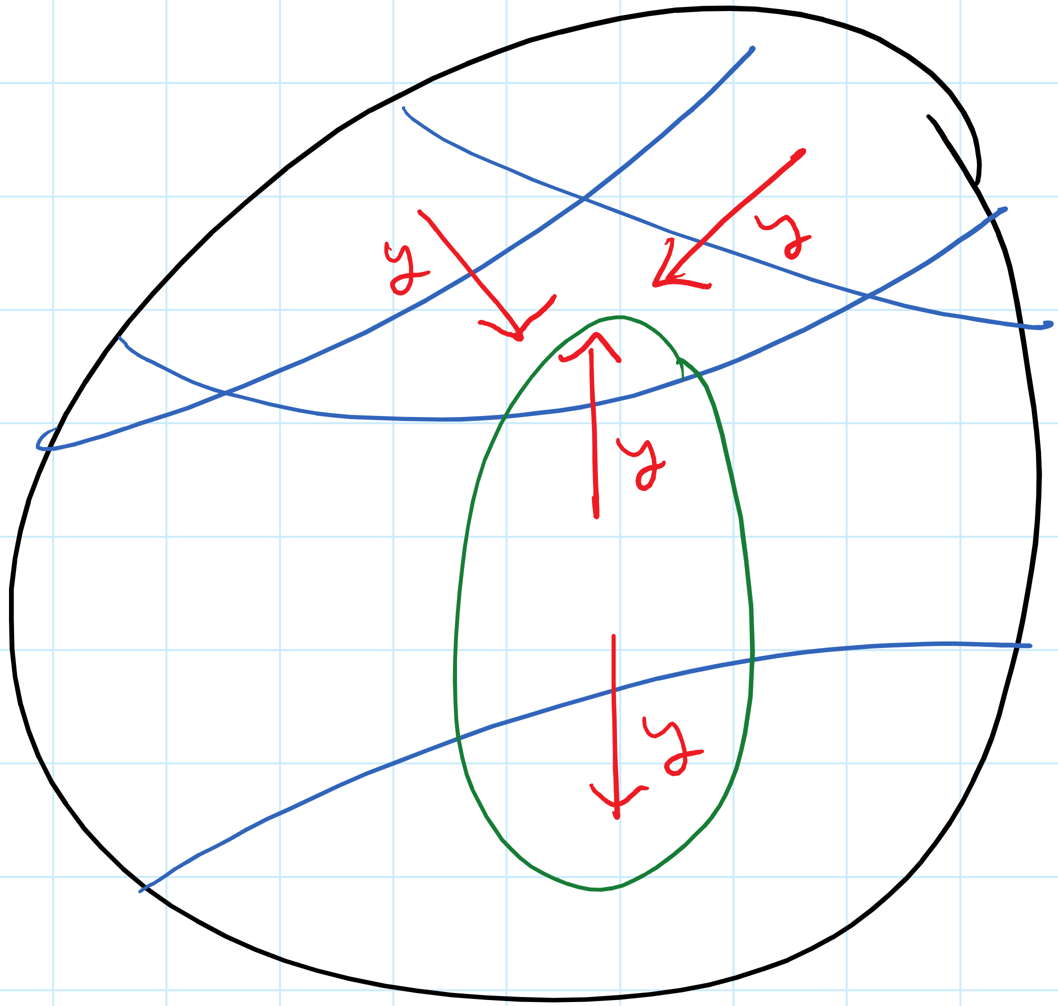

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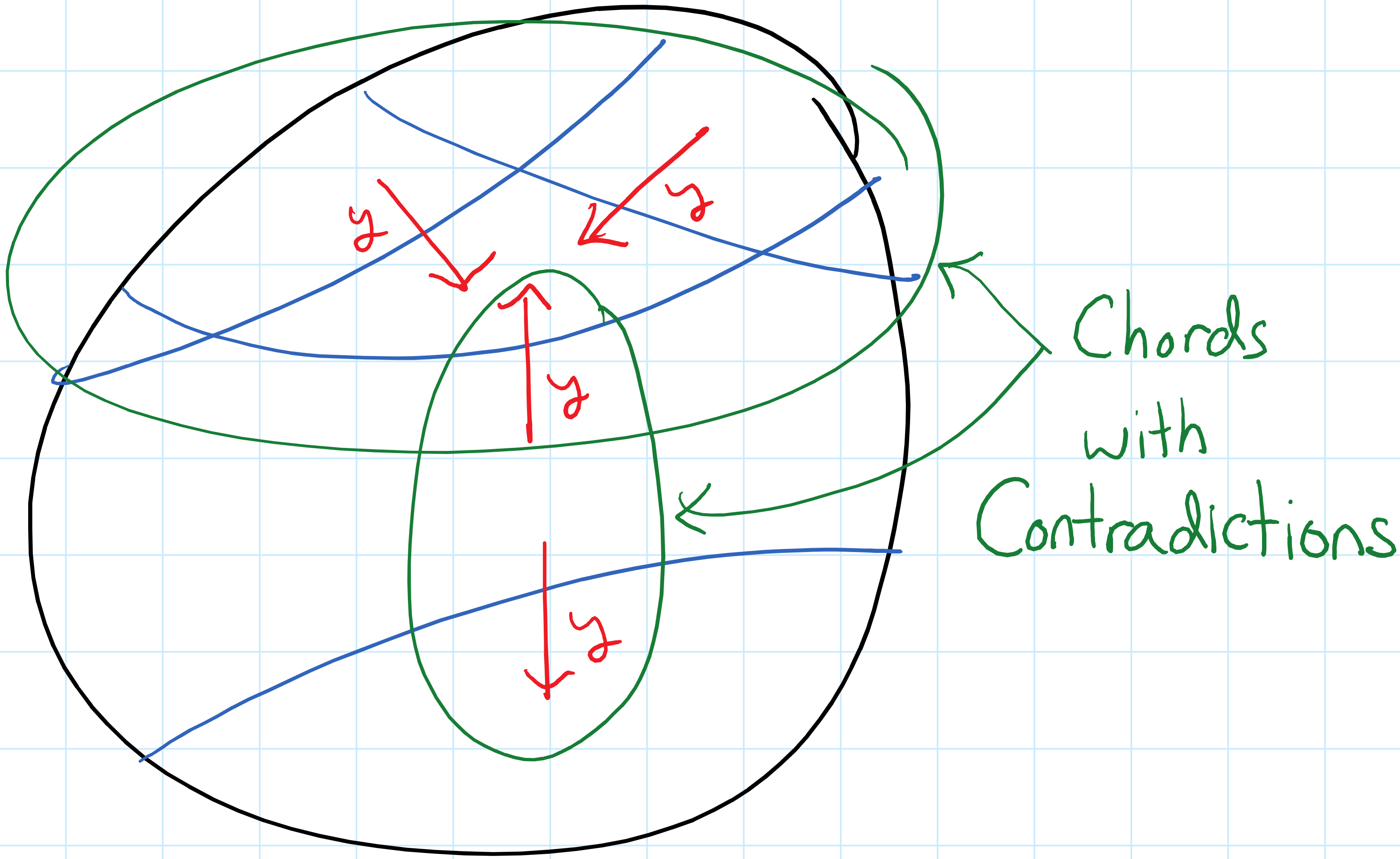
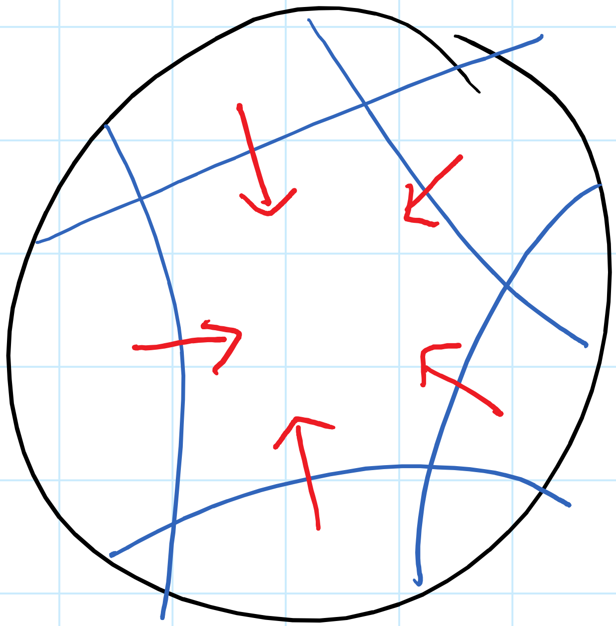
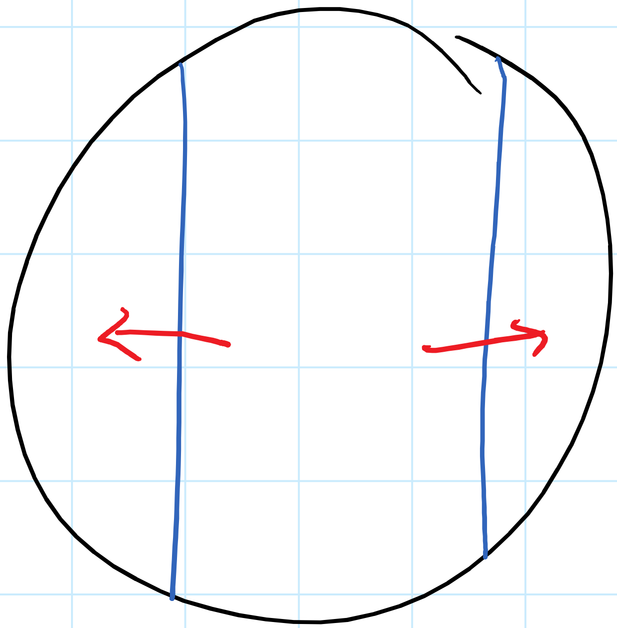


Now if there are chords with contradictions we can find a minimal contradictory set:

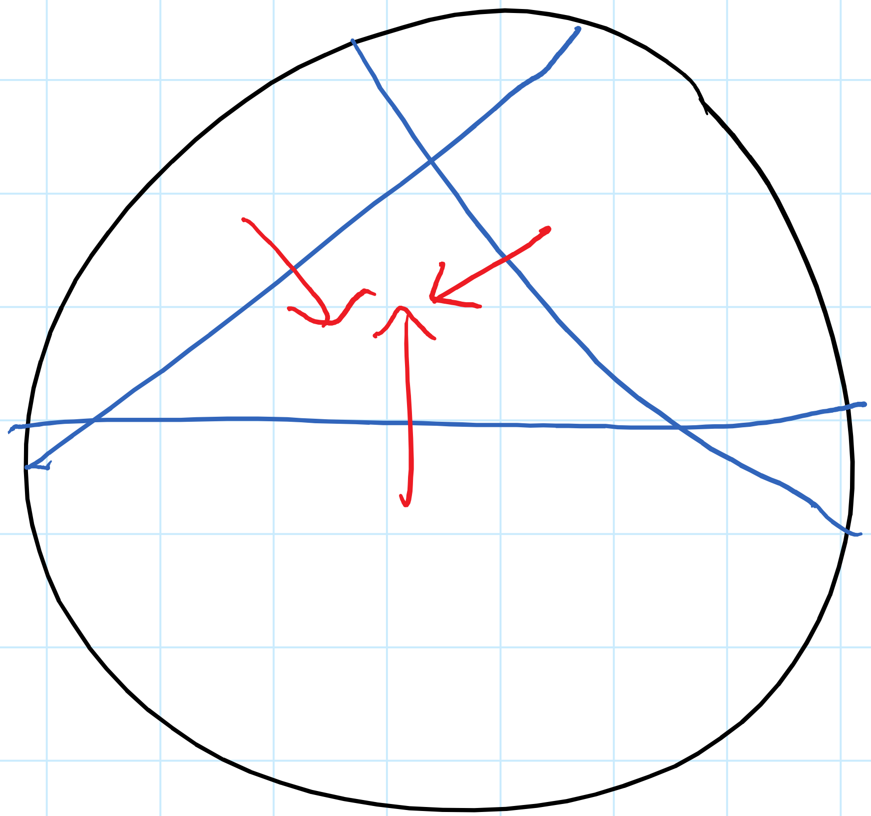


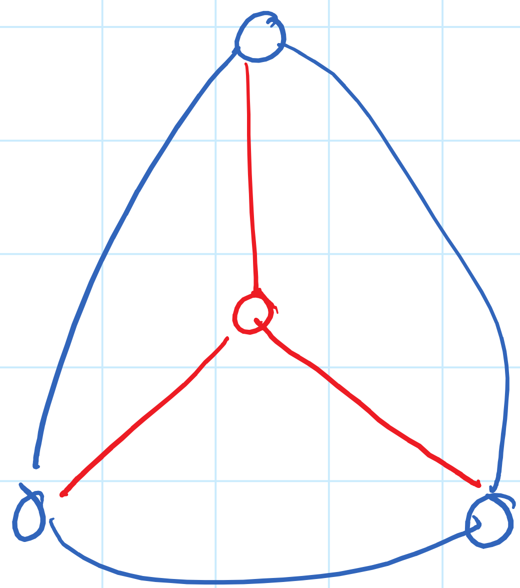
Induced C_k, $k \geq 3$



Two non-adjacent vertices

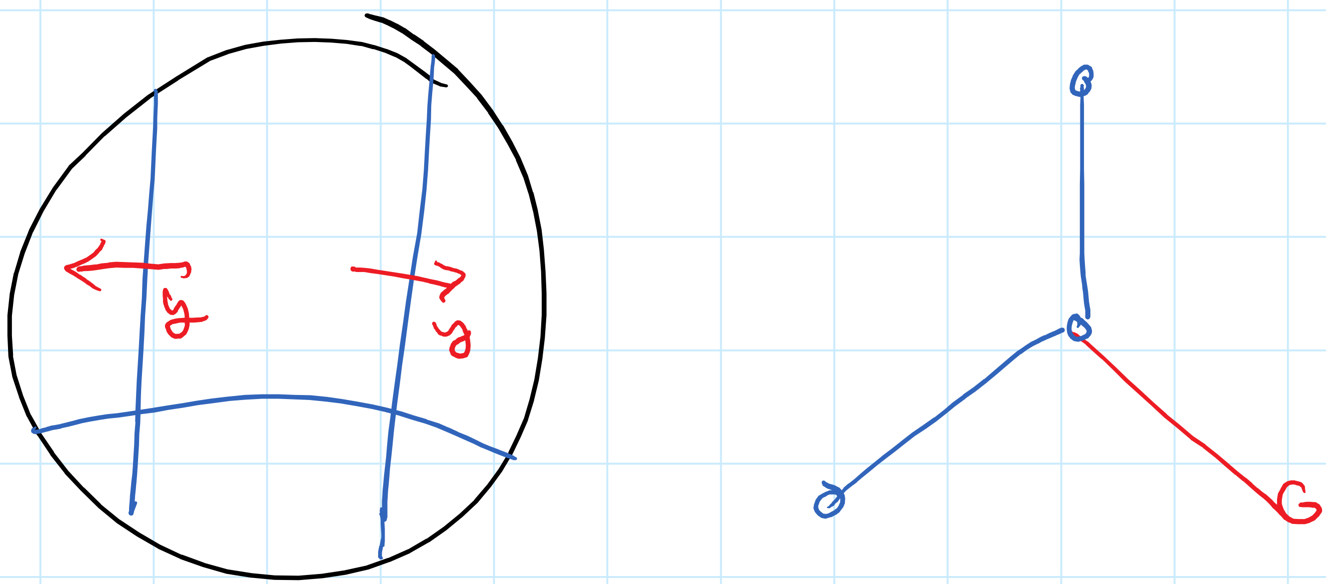
- If $k \geq 5$ then G violates (N)
- So $C_k \hookrightarrow C_3 +$ axle $= K_4$

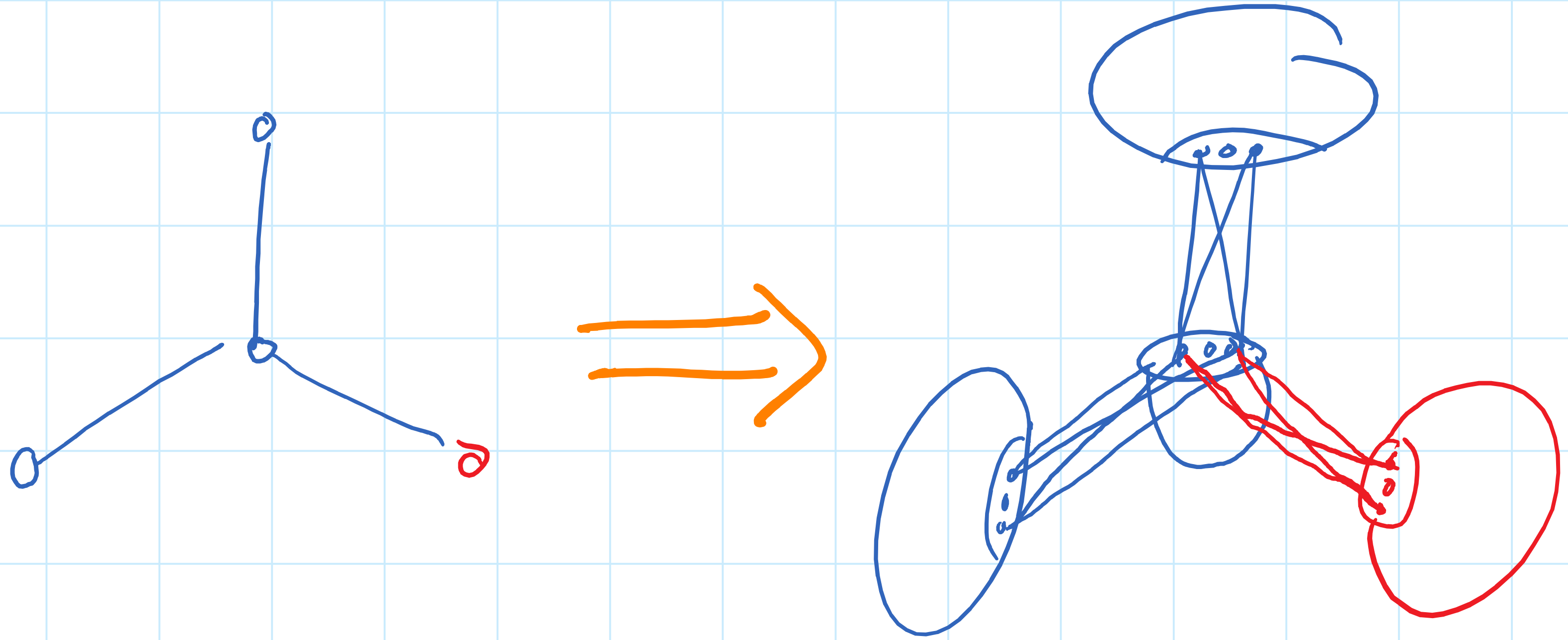


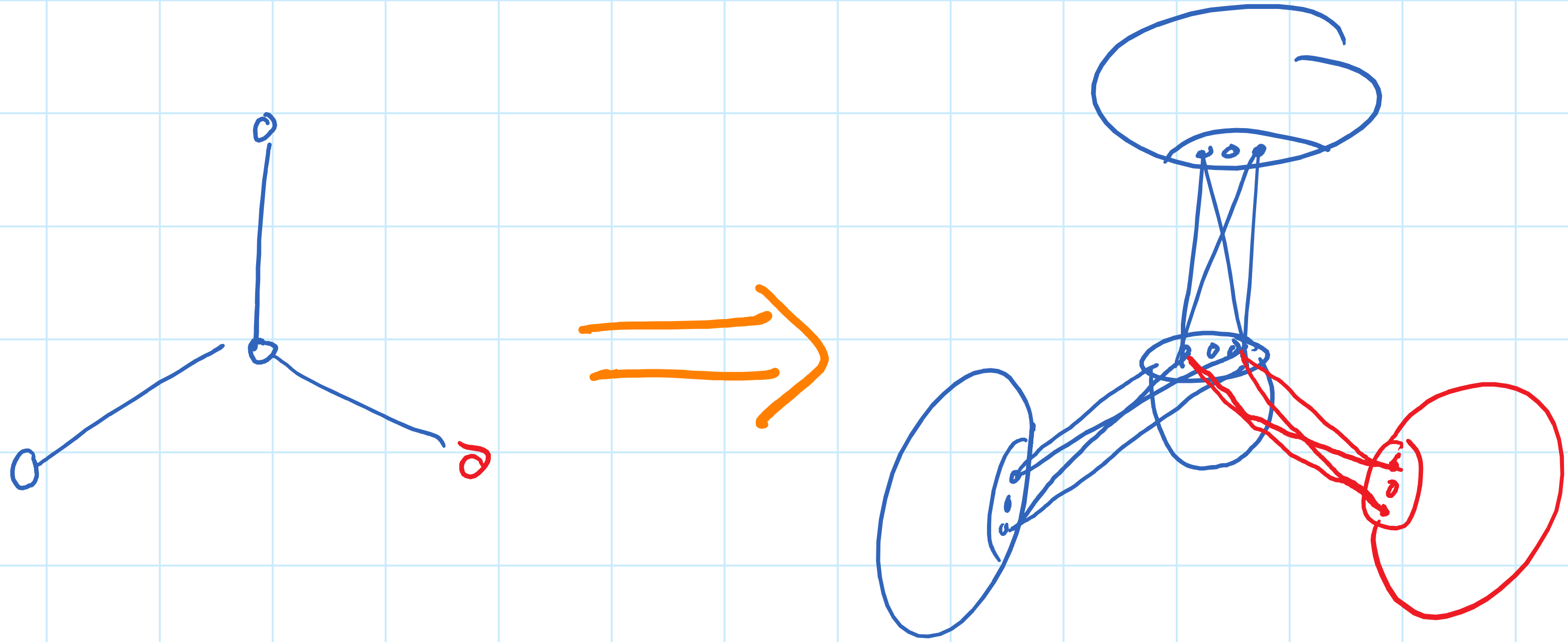


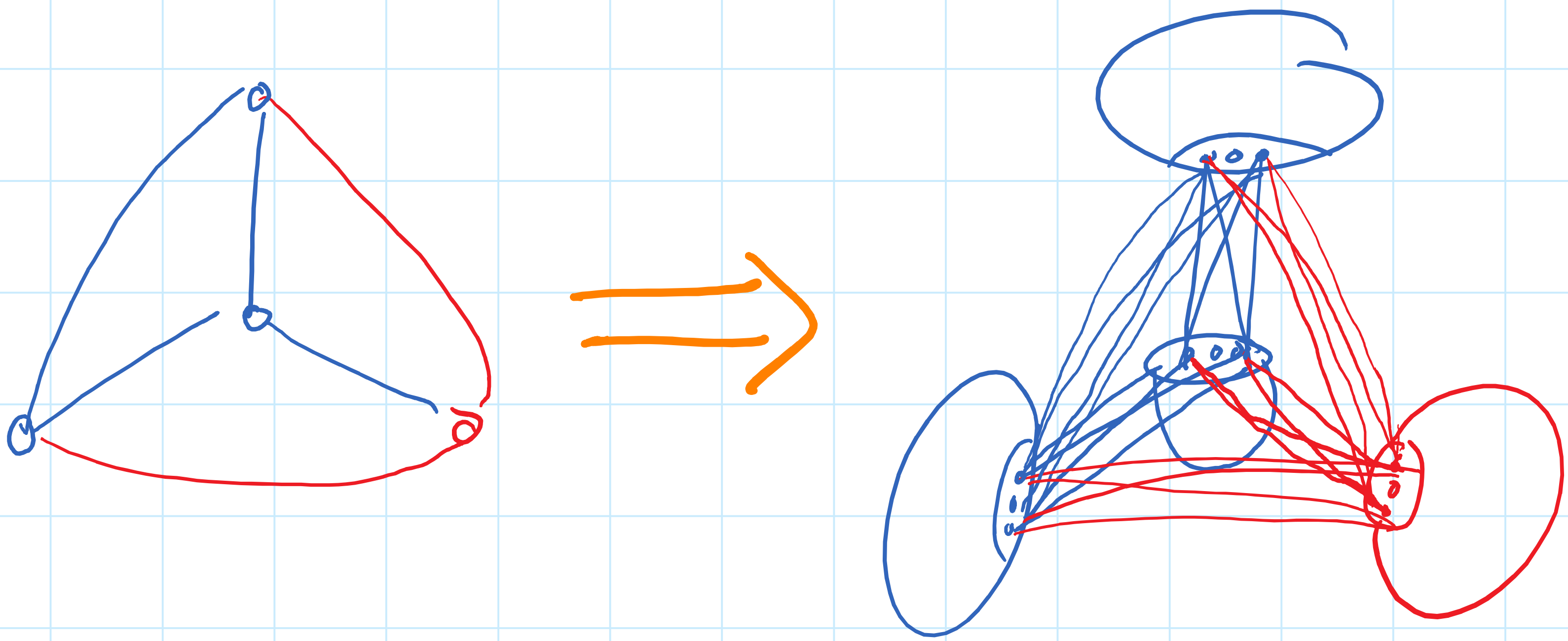
- As the chord diagram is connected there is a path between 2-non-adjacent vertices

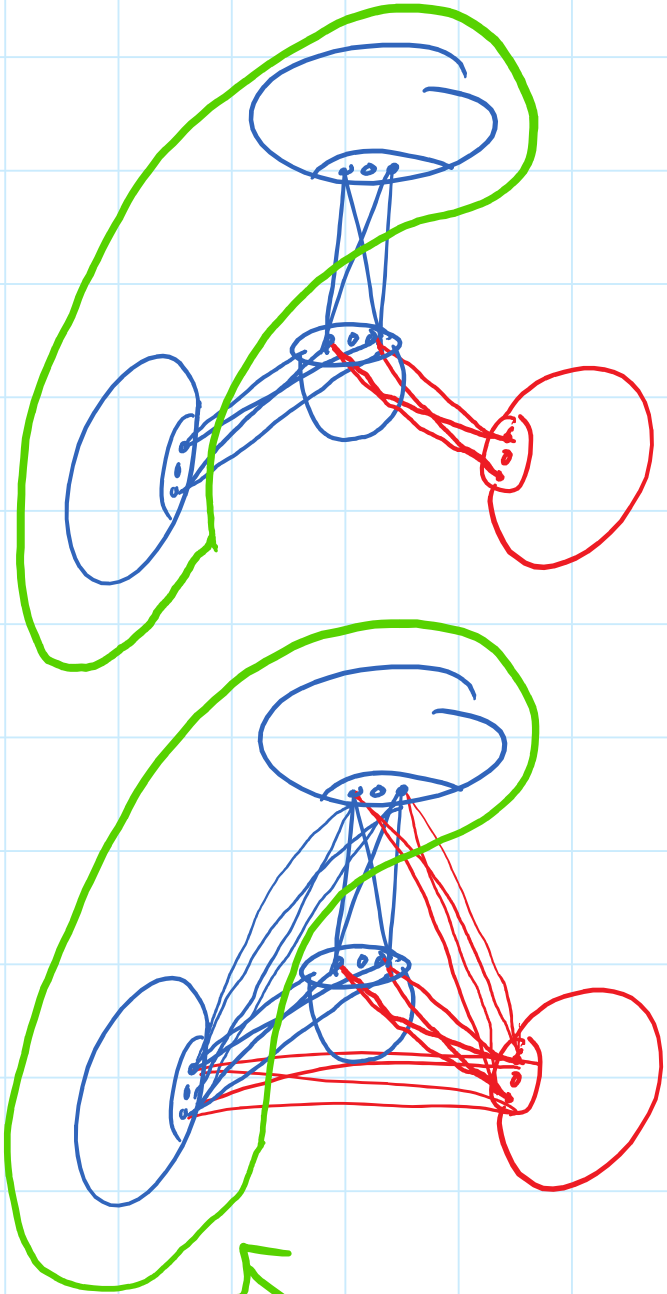
- So one gets (by minimality):



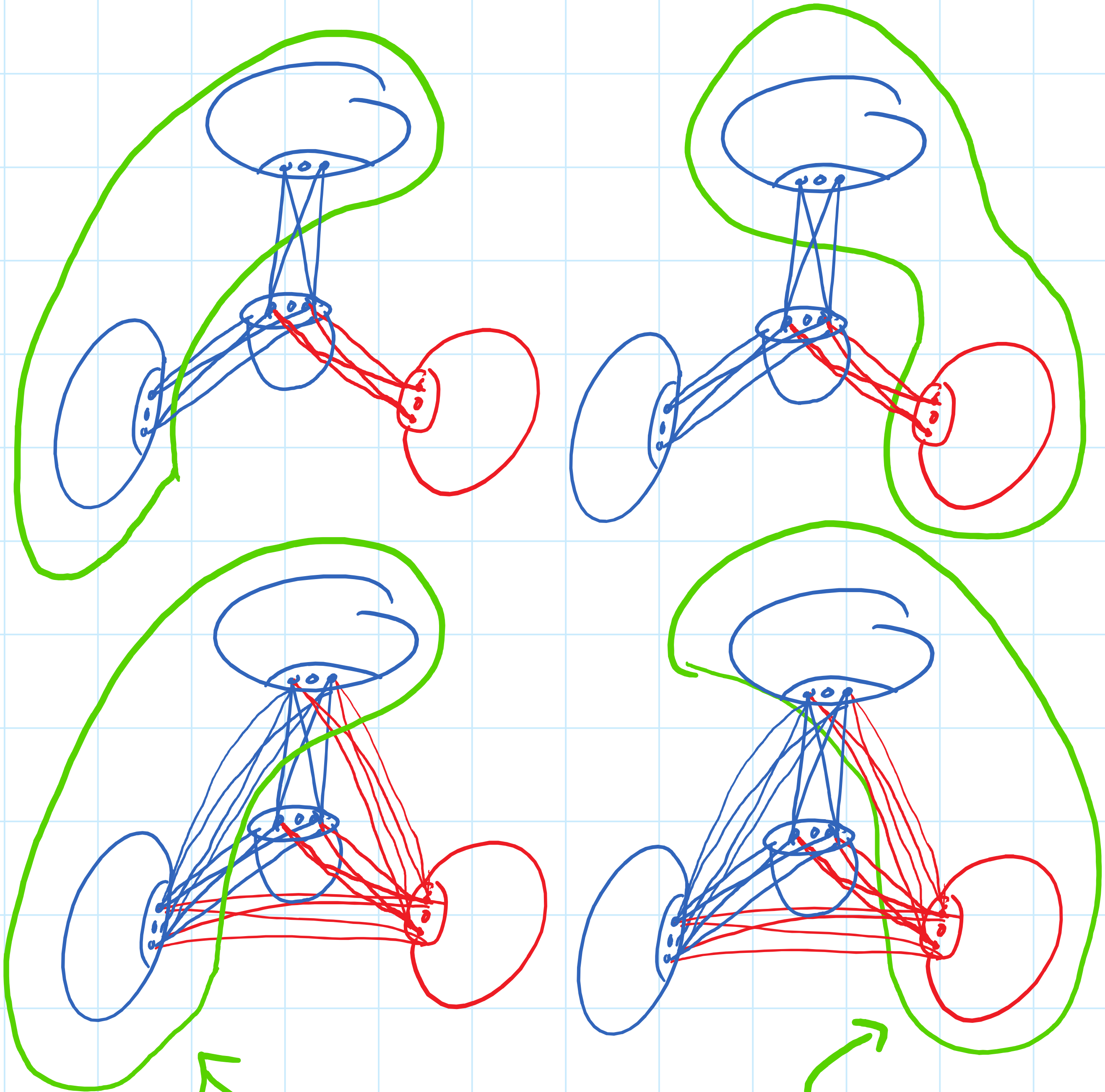




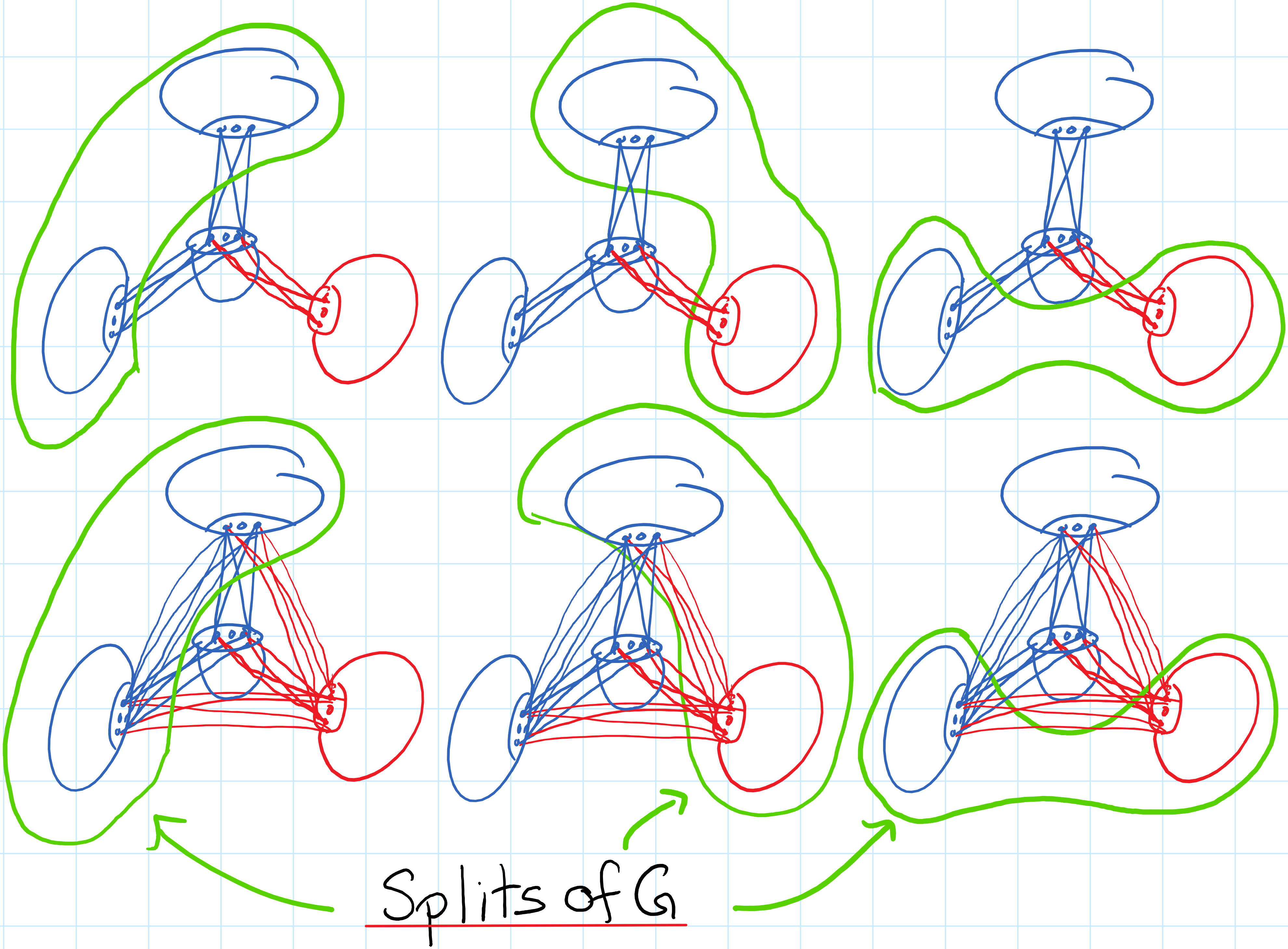




Splits of G



Splits of G


Splits of G