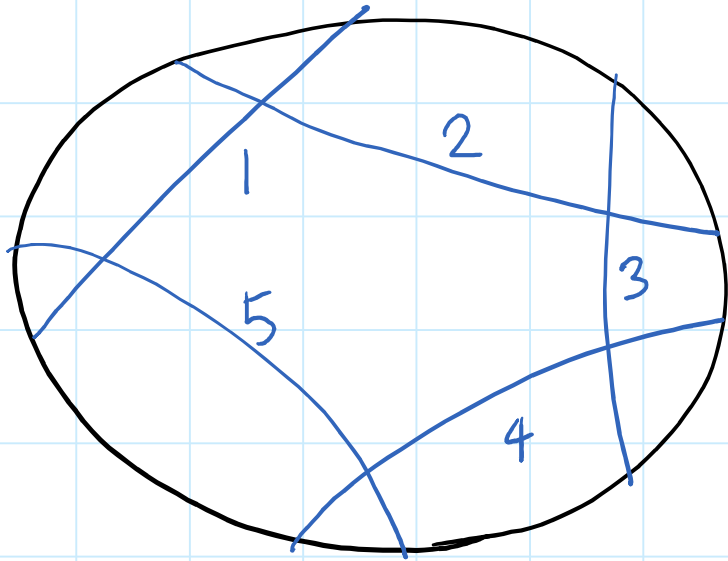


# Recognizing Circle Graphs

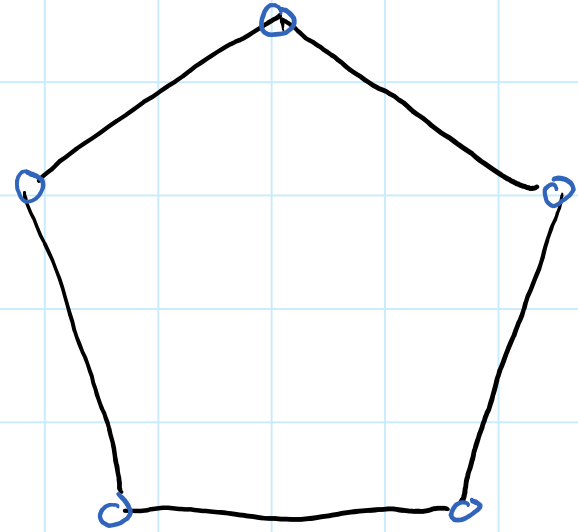
Jim Geelen

Edward Lee

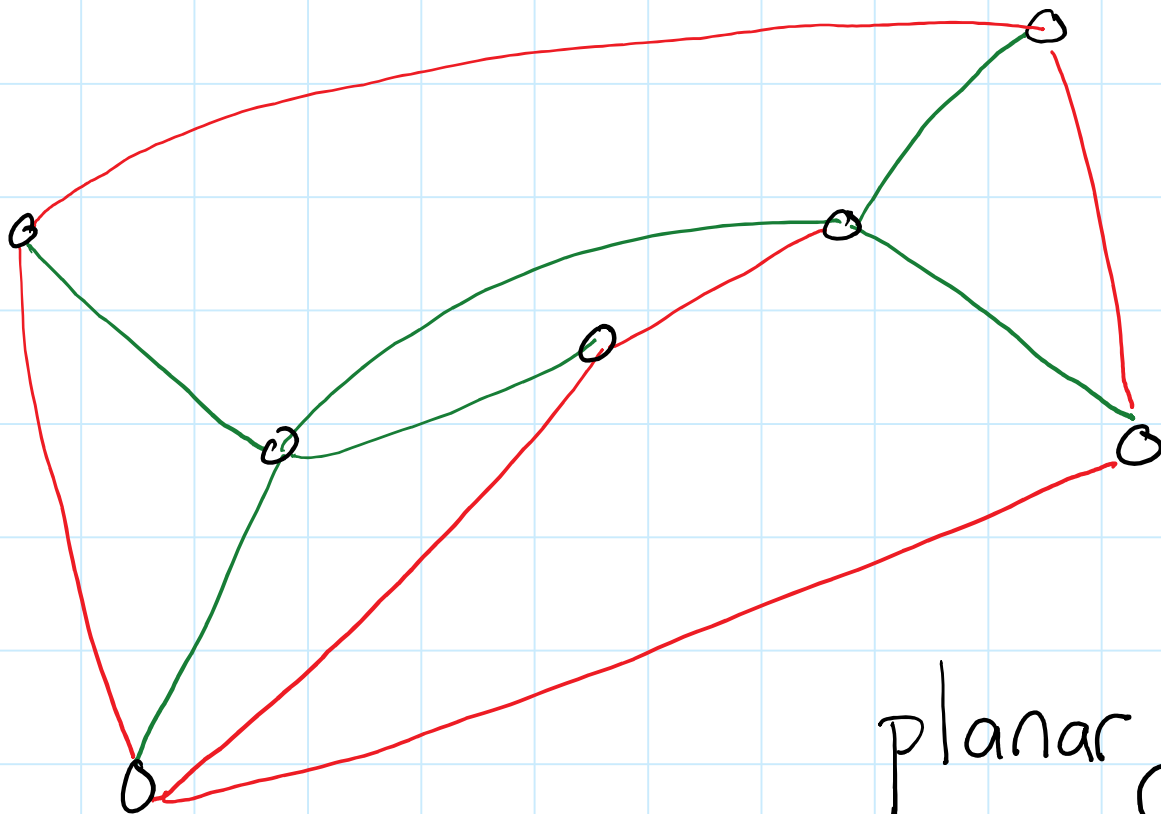
University of Waterloo



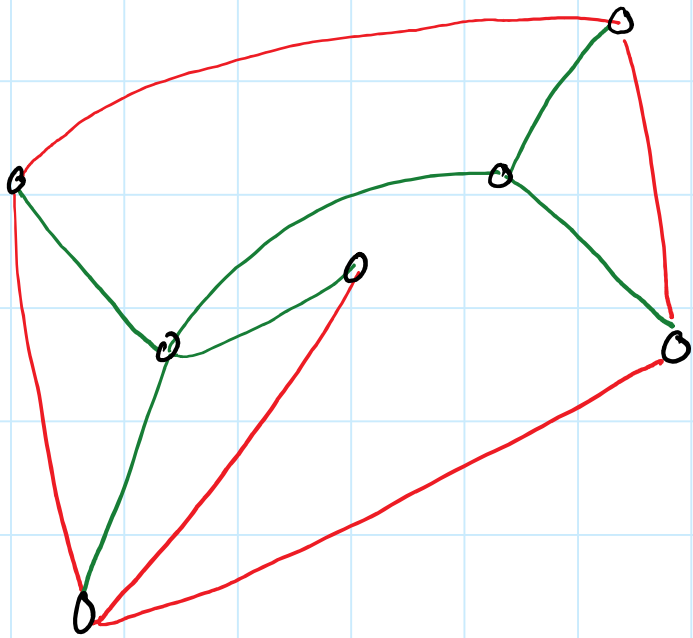
Chord Diagram  
 $\mathcal{C}$

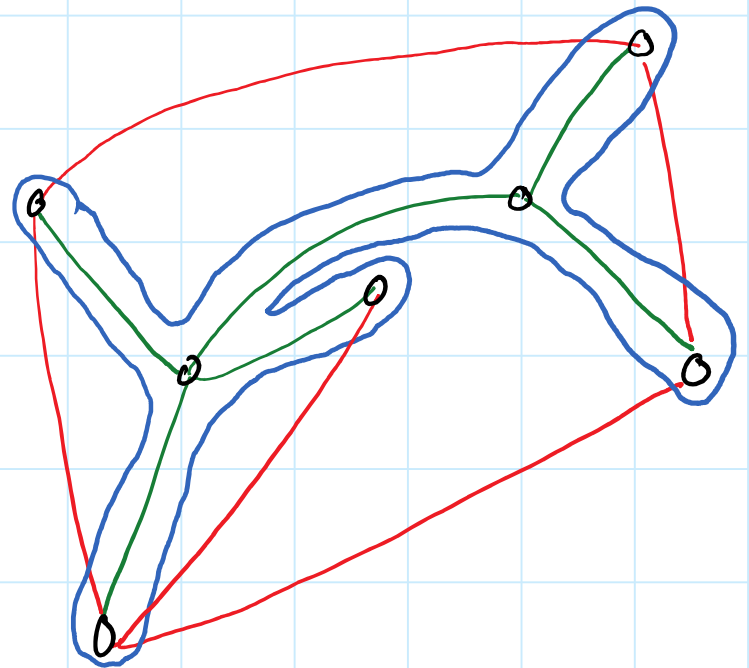
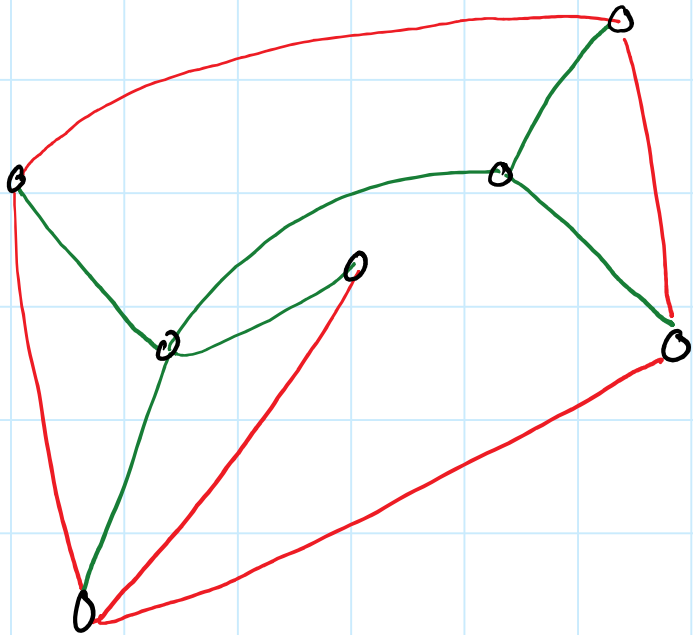


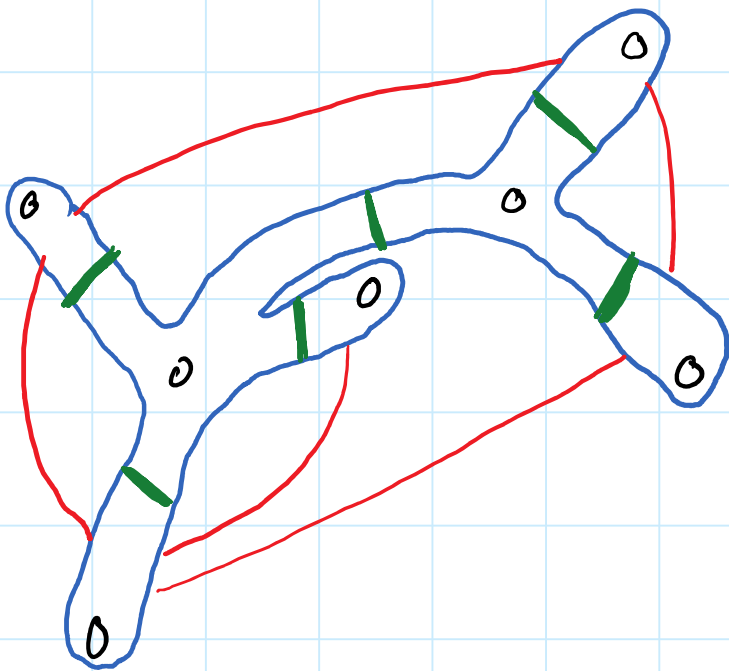
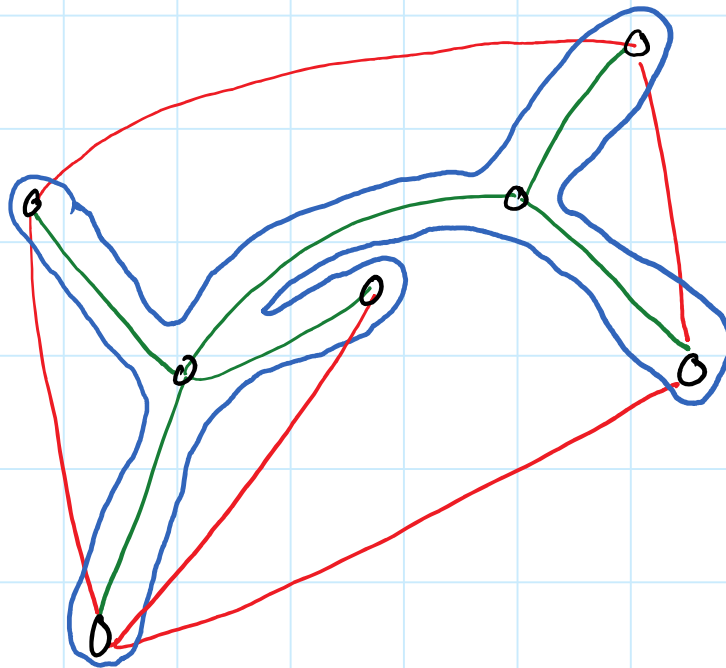
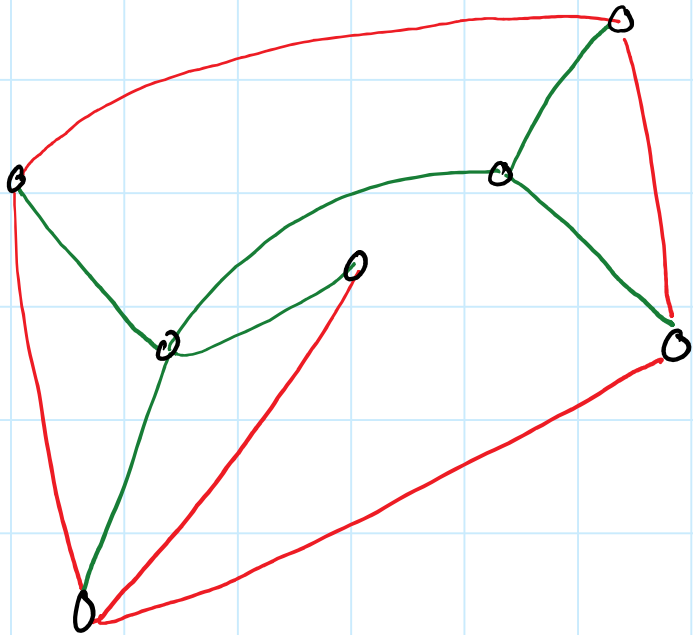
Circle Graph  
 $G$

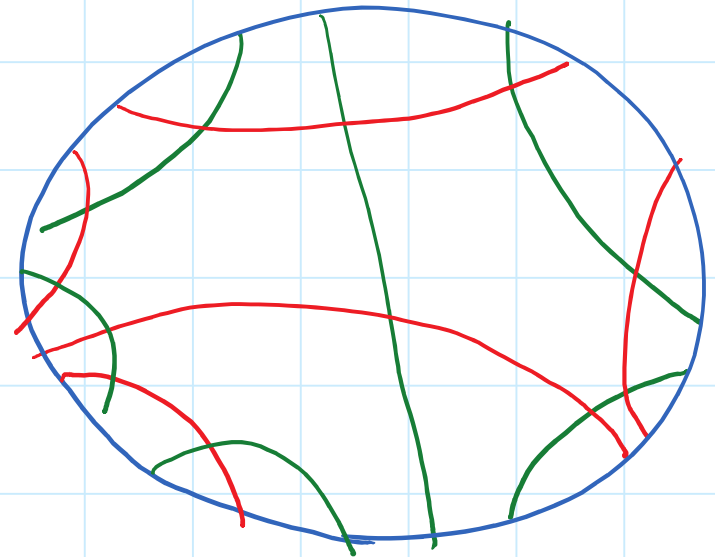
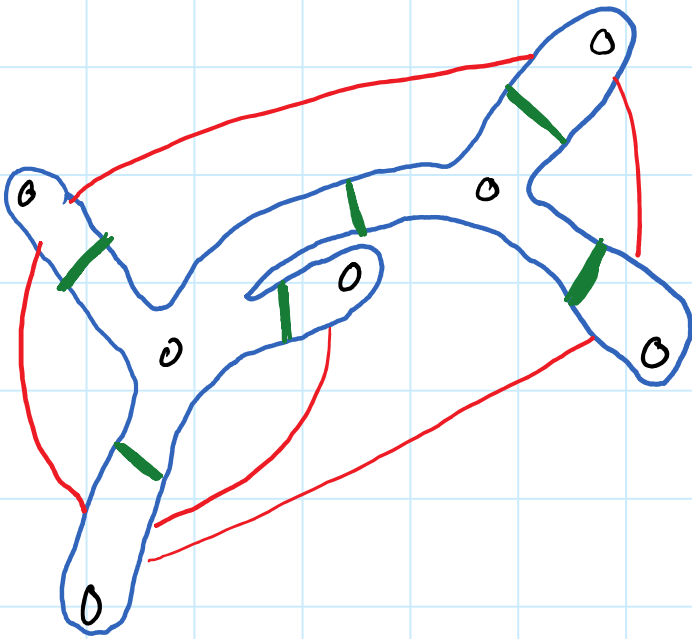
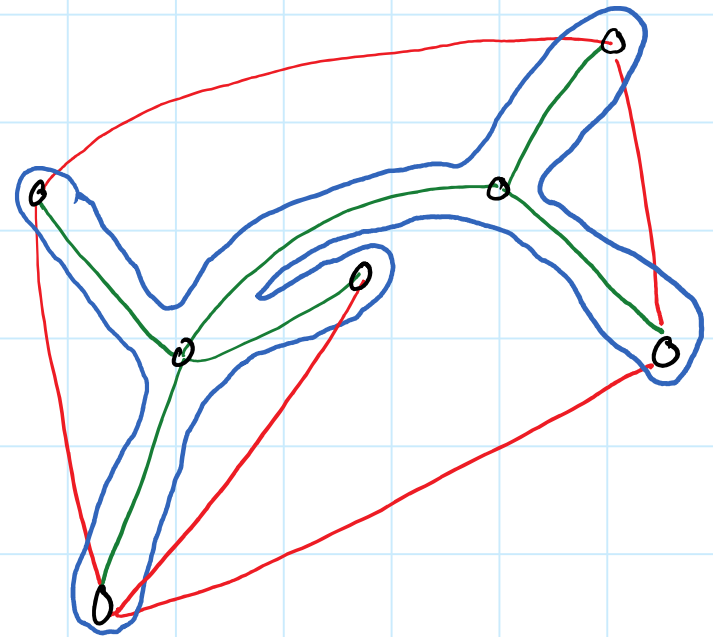
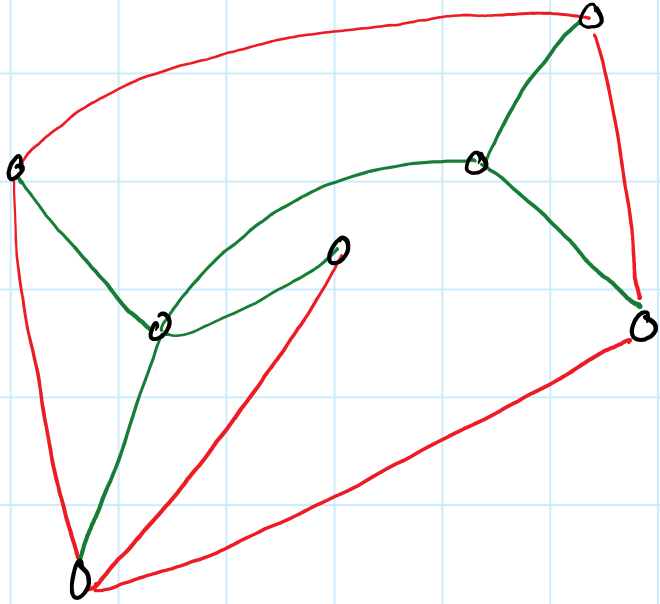


planar graph  
green - spanning tree  
red - everything else

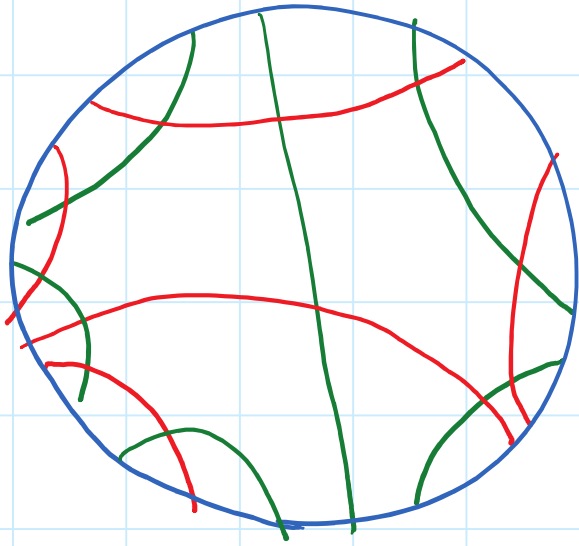








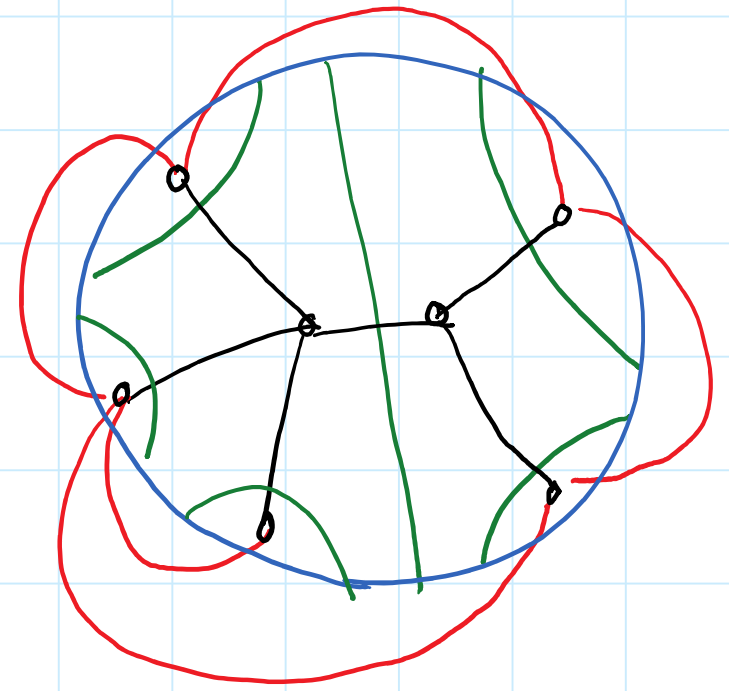
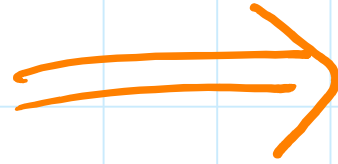
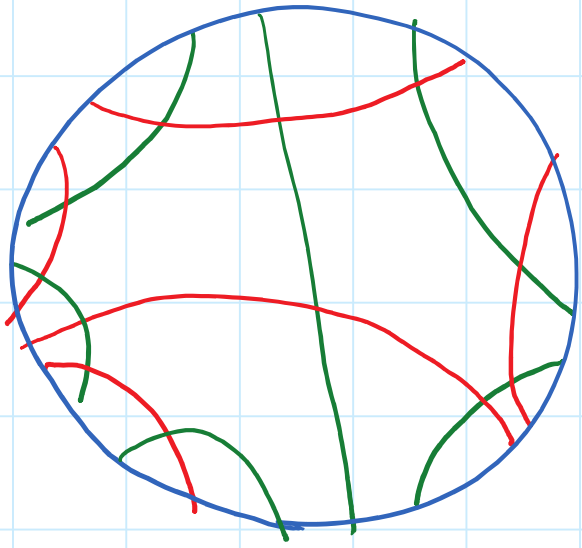
bipartite chord diagram



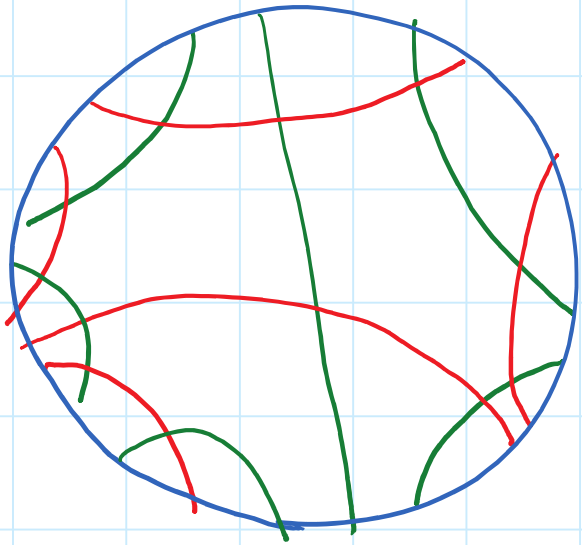
bipartite chord diagram

This process is also reversible.

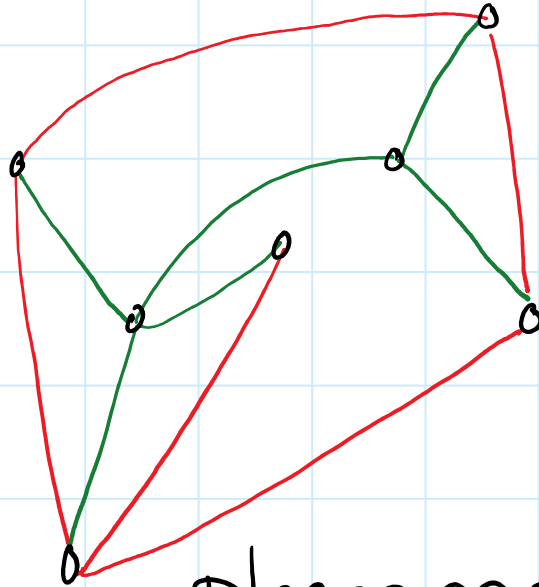
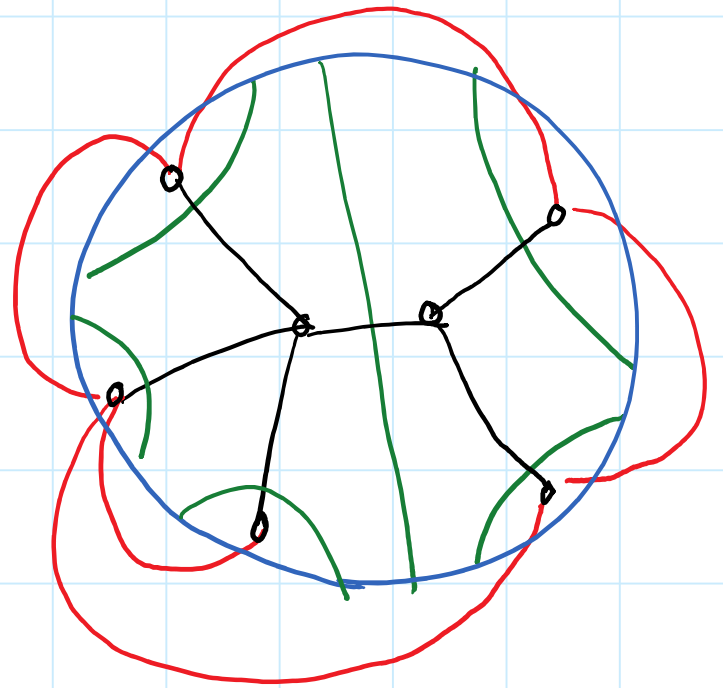
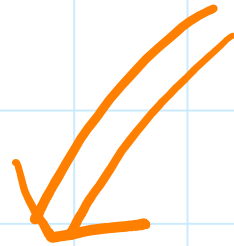
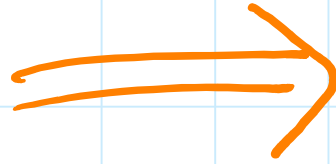




bipartite chord diagram



bipartite chord diagram



planar graph

This construction is known as

## De Frayessix's Theorem

The important takeaway is that:

Characterizations of circle graphs



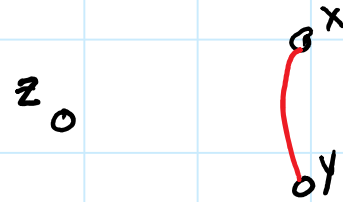
Characterizations of planar graphs

# Naji System for $G$ :

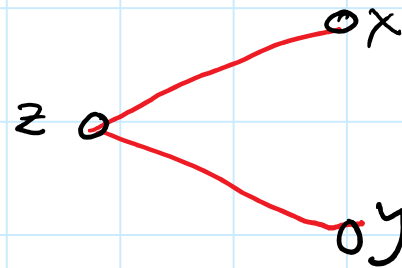
(1)  $\beta(x,y) + \beta(y,x) = 1$  for



(2)  $\beta(z,x) + \beta(z,y) = 0$  for

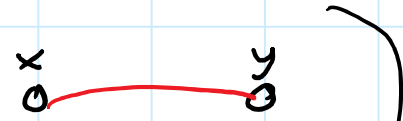


(3)  $\beta(z,x) + \beta(z,y) + \beta(x,y) + \beta(y,x) = 1$  for

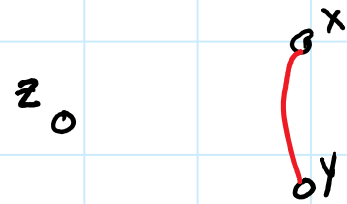


# Naji System for $G$ :

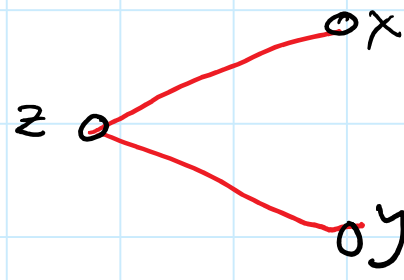
(1)  $\beta(x,y) + \beta(y,x) = 1$  for



(2)  $\beta(z,x) + \beta(z,y) = 0$  for



(3)  $\beta(z,x) + \beta(z,y) + \beta(x,y) + \beta(y,x) = 1$  for



(N)

Theorem (Naji)  $G$  is a circle graph  $\Leftrightarrow$  (N) has a solution over  $GF(2)$ .

# An interlude into complexity:

- (N) has  $n^2$  variables  
 $\sim n^3$  equations

$$\begin{bmatrix} x & & & \\ & x & & \\ & & \ddots & \\ & & & x \end{bmatrix} = \begin{bmatrix} x \\ \vdots \\ x \end{bmatrix}$$

The diagram shows a square matrix with a diagonal of 'x' variables. A brace above the matrix is labeled  $n^2$ , and a brace to the left is labeled  $n^3$ . To the right of the matrix is an equals sign followed by a vertical vector of 'x' variables.

## An interlude into complexity:

- (N) has  $n^2$  variables  
     $\sim n^3$  equations

A diagram illustrating a matrix equation. On the left, a large square matrix is enclosed in red brackets. A horizontal brace above the matrix is labeled  $n^2$ , and a vertical brace to its left is labeled  $n^3$ . Inside the matrix, there are several 'x' characters and a diagonal line of dots, representing a sparse matrix. To the right of the matrix is an equals sign followed by a vertical vector of red brackets containing several 'x' characters and dots, representing a vector of size  $n^2$ .

- Gaussian (column) elimination can be done in  $O(n^5)$  time for each column.

## An interlude into complexity:

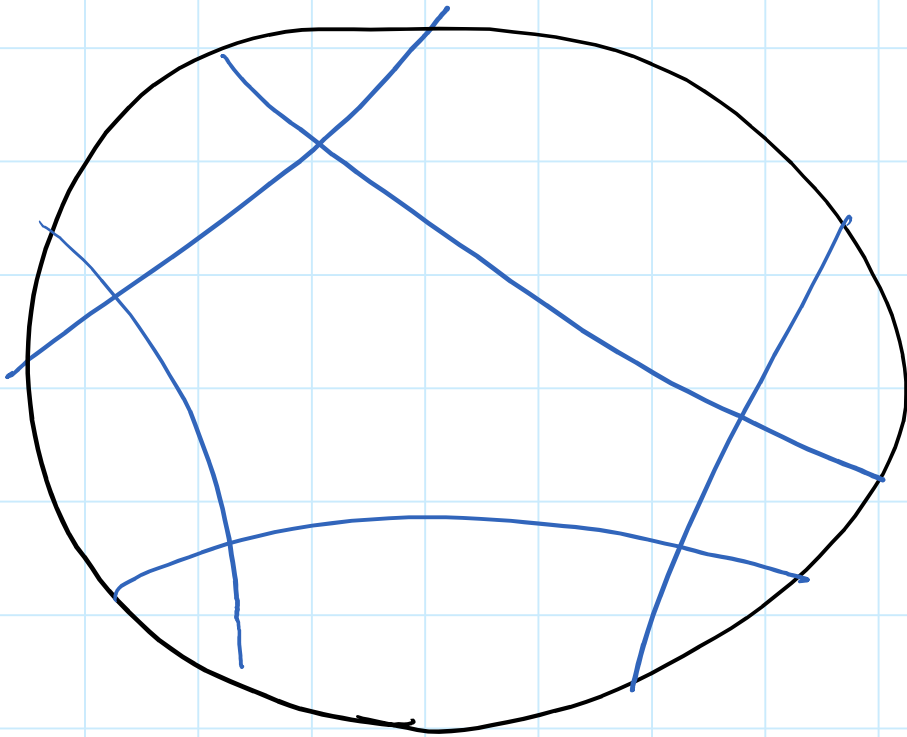
- $(N)$  has  $n^2$  variables  
 $\sim n^3$  equations

$$\begin{bmatrix} x & & & \\ & x & & \\ & & \ddots & \\ & & & x \end{bmatrix} = \begin{bmatrix} x \\ \vdots \\ x \end{bmatrix}$$

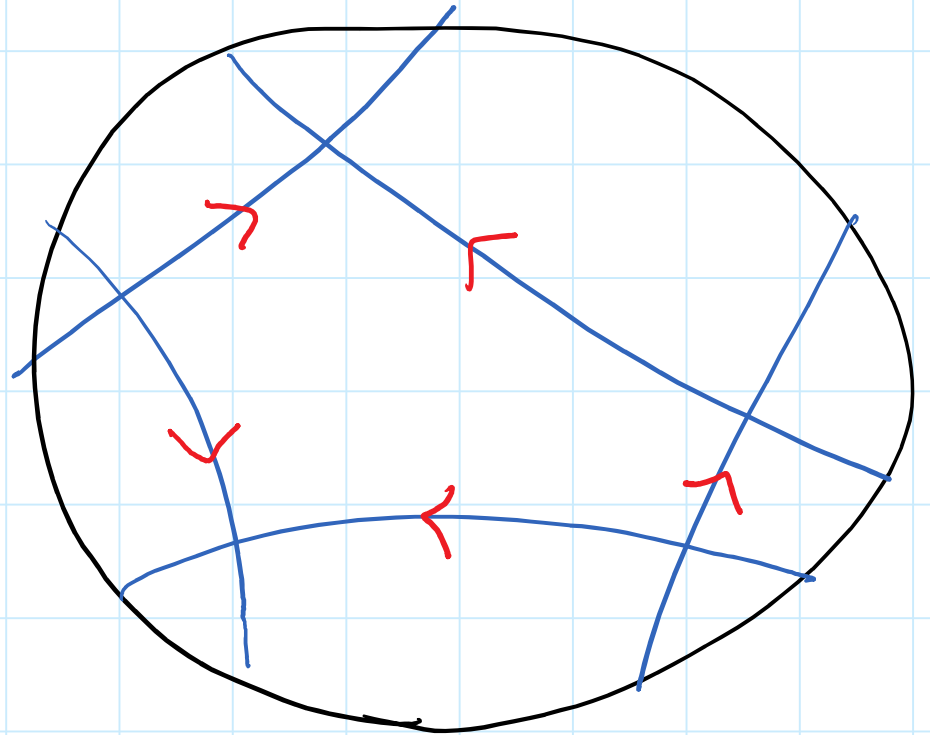
- Gaussian (column) elimination can be done in  $O(n^5)$  time for each column.
- So  $O(n^7)$  algorithm for recognizing circle graphs.



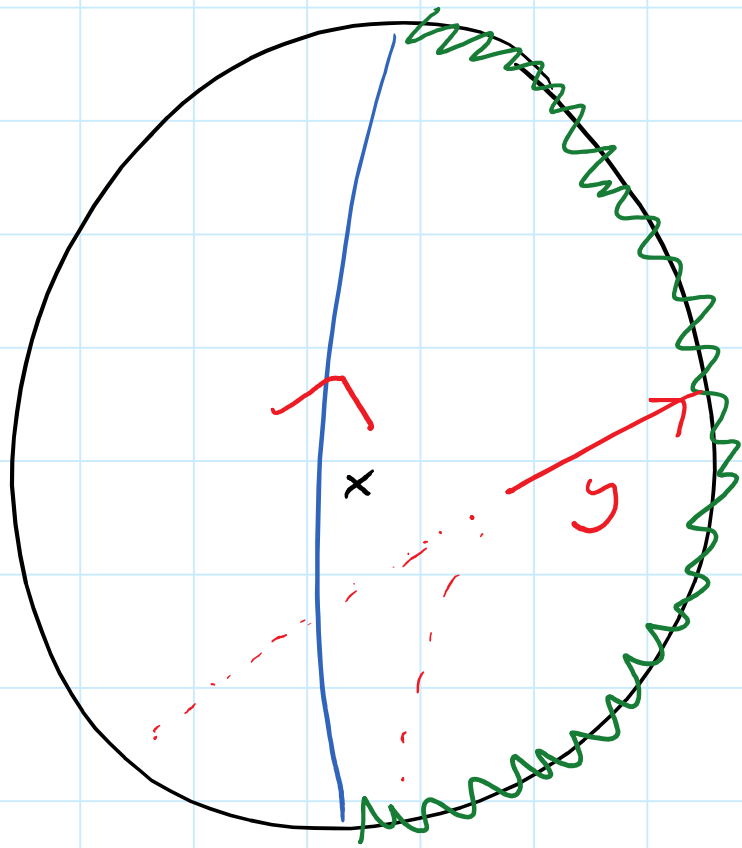
# Circle Graph $\implies$ Solution



chord diagram



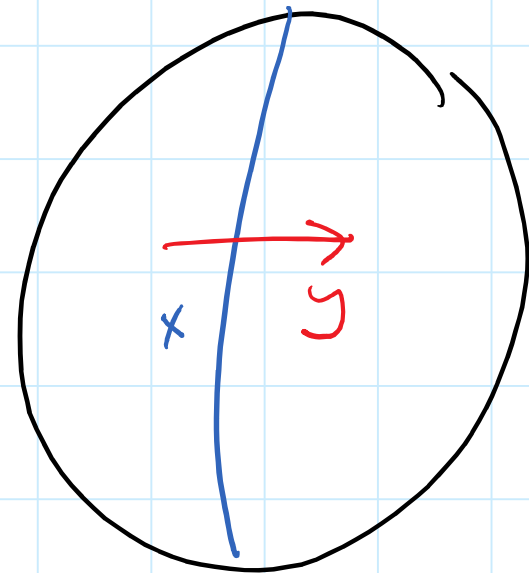
orientations



Take  $\beta(x, y) = 0$  if the head of  $y$  is to the right of  $x$ .

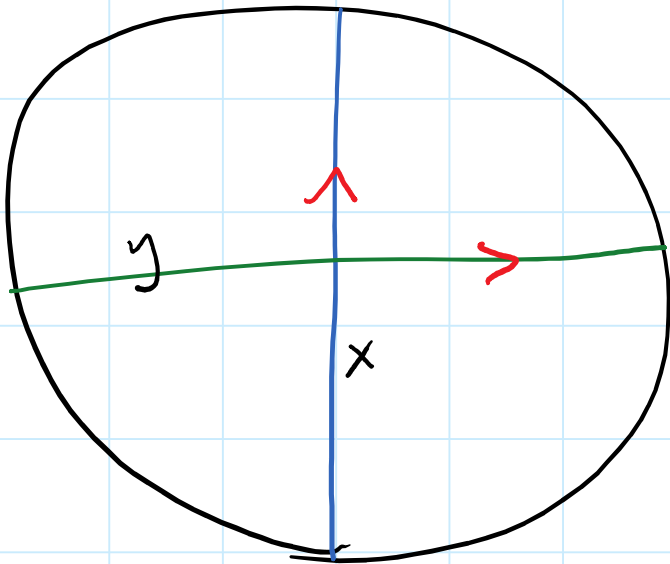
Notation:  $\beta(\mathcal{C})$  for such solutions.

Aside: will often draw  
as:



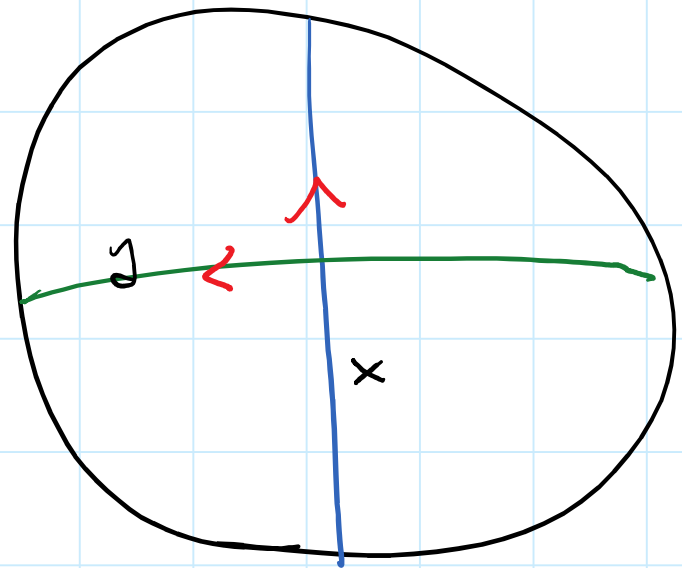
Equation (1):  $x \overset{\text{red arc}}{\longleftrightarrow} y$

$$P(x,y) + P(y,x) = 1$$



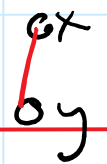
$$P(x,y) = 1$$

$$P(y,x) = 0$$

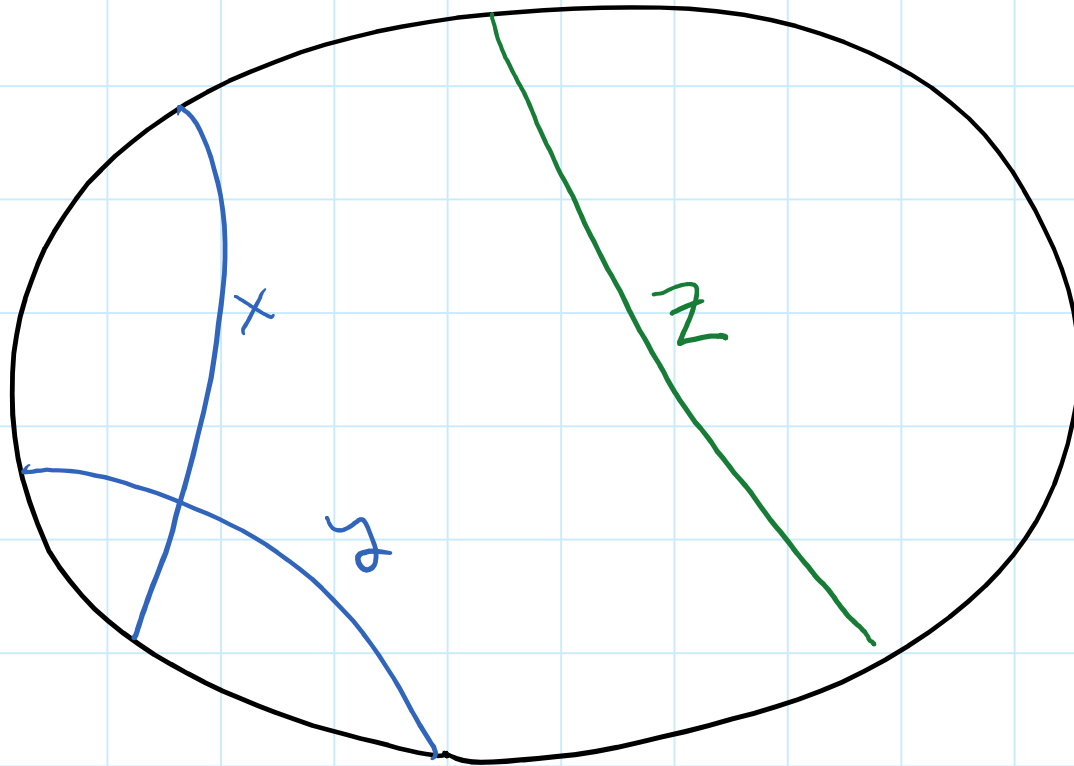


$$P(y,x) = 1$$

$$P(x,y) = 0$$

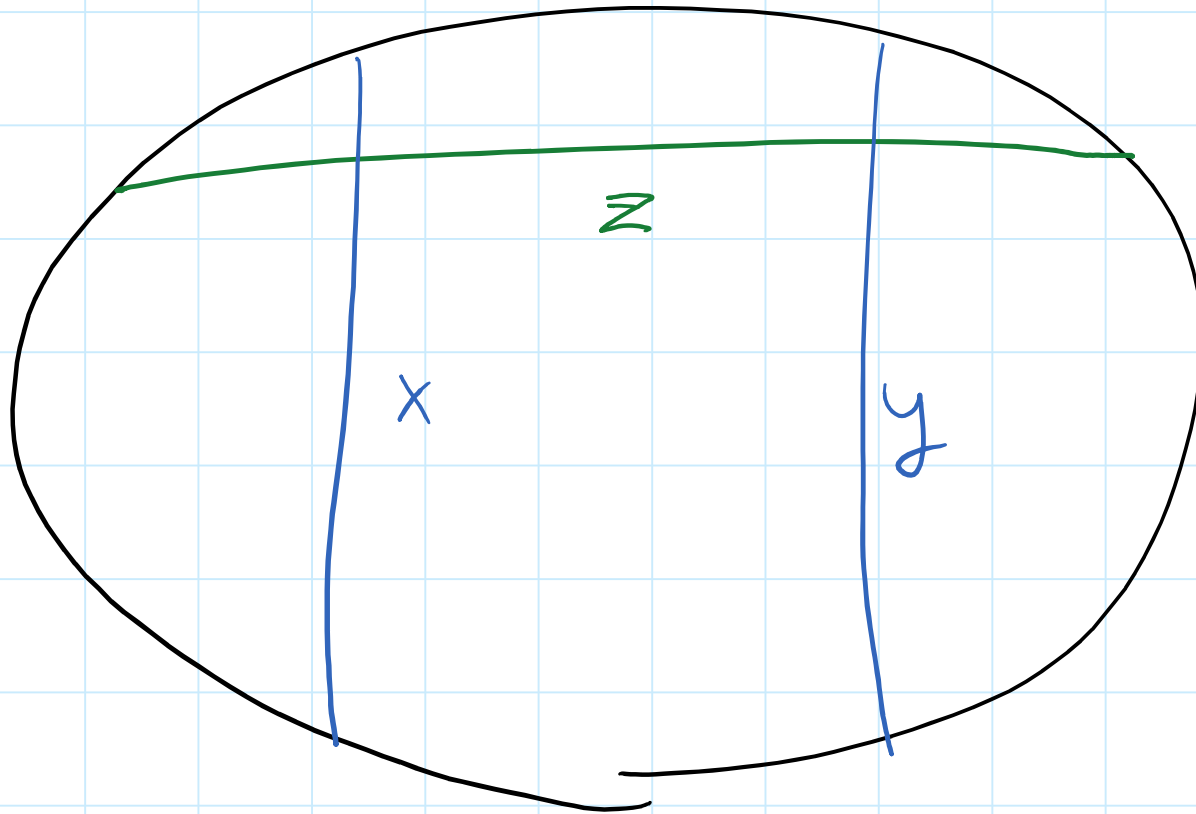
Equation (2):  $z \circ$  

$$\beta(z, x) + \beta(z, y) = 0$$



Equation (3):  $z^a$   $\begin{matrix} \nearrow^{o_x} \\ \searrow^{o_y} \end{matrix}$

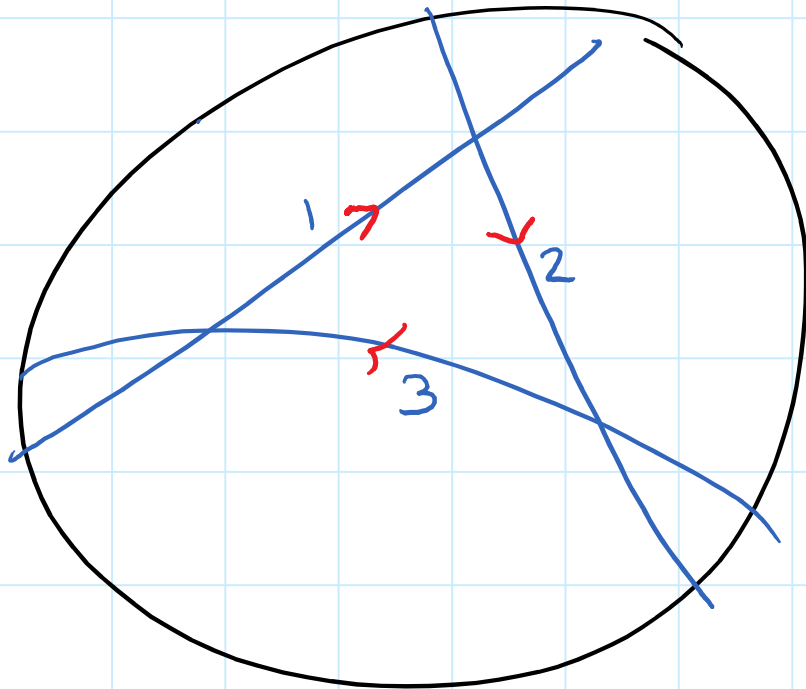
$$\beta(z, x) + \beta(z, y) + \beta(x, y) + \beta(y, x) = 1$$

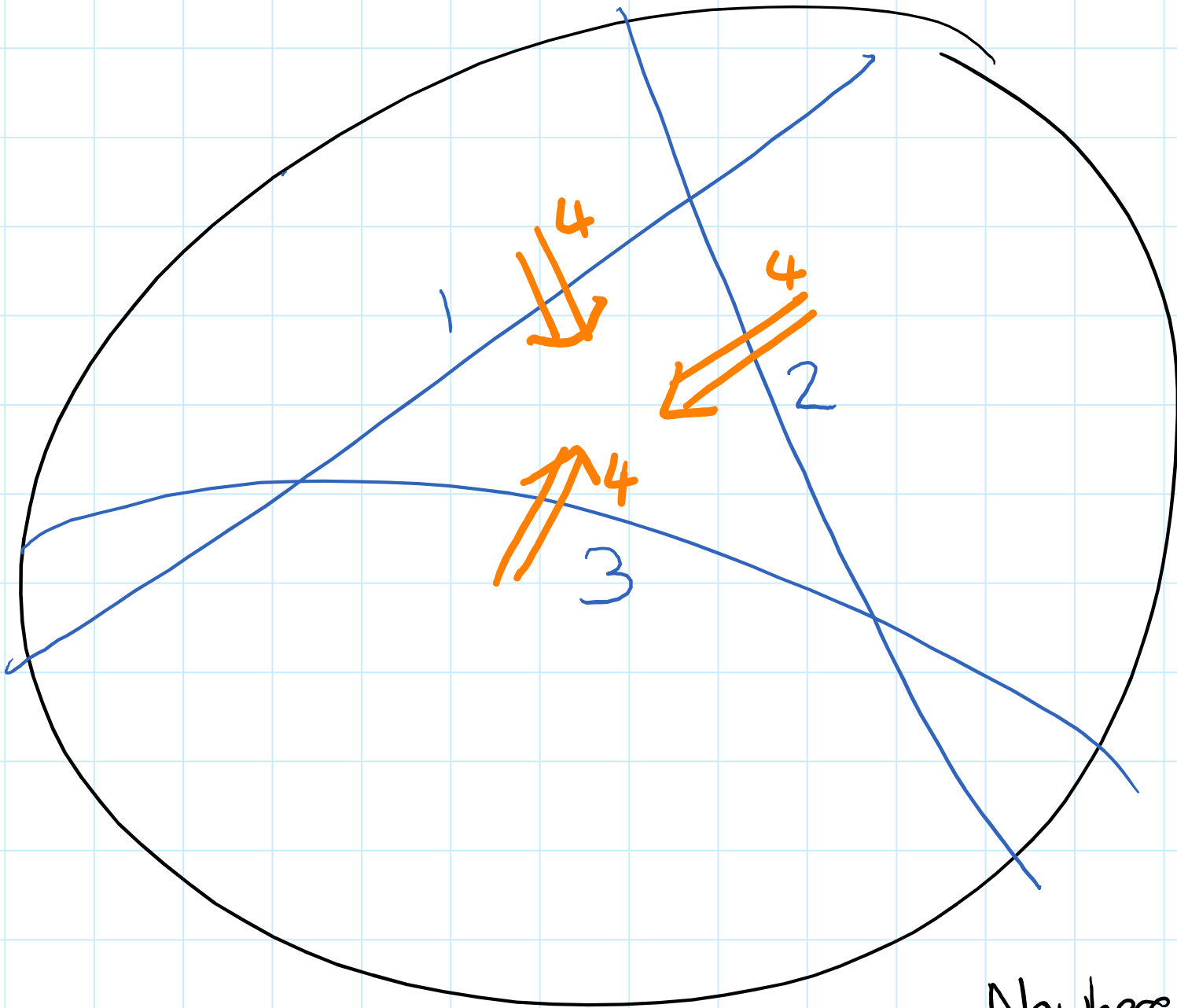


Not all Naji solutions are so natural.

Consider the following solution to  $K_4$ :

$$\begin{array}{l|l} \beta(1,2) = \beta(2,3) = \beta(3,1) = 0 & \text{everything else } 1. \\ \beta(1,4) = \beta(2,4) = \beta(3,4) = 0 & \end{array}$$





Nowhere to put 4.

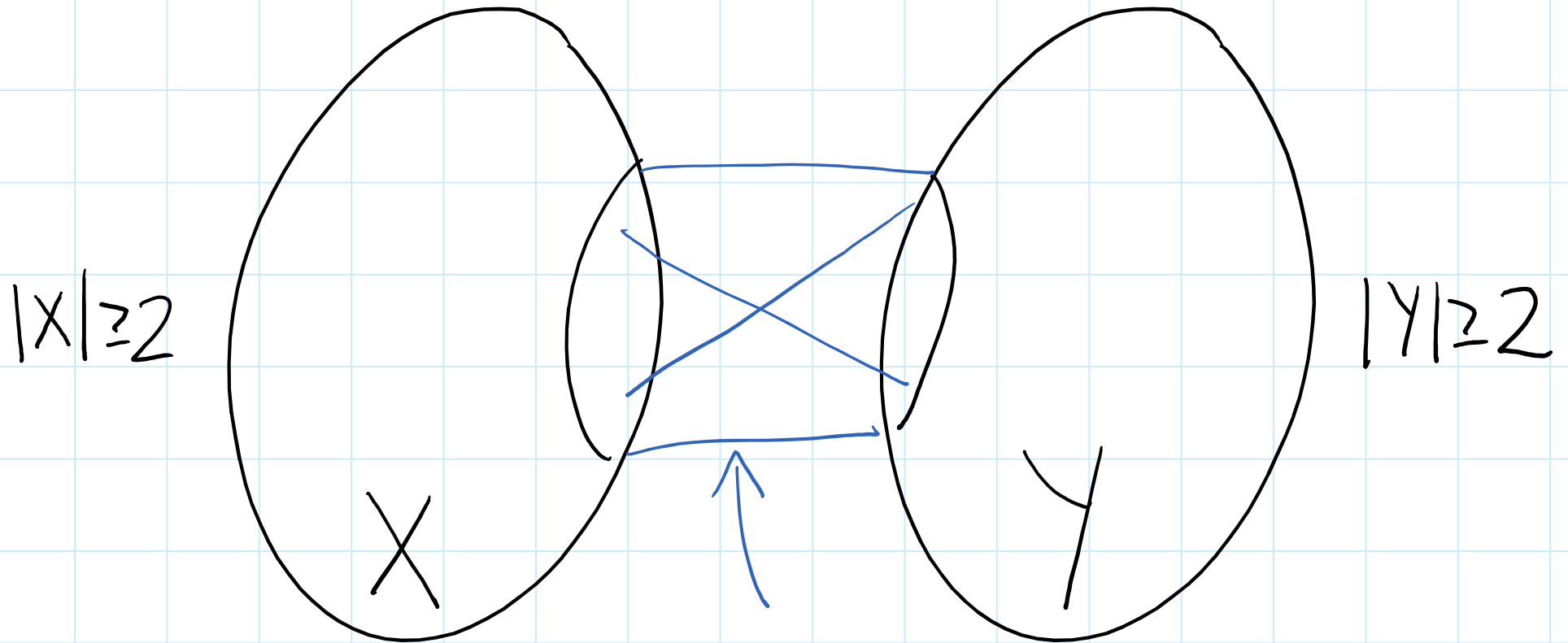
$$A\bar{x} = A\bar{y} = A\bar{z} = b$$

$$\Rightarrow A(\bar{x} + \bar{y} + \bar{z}) = b$$

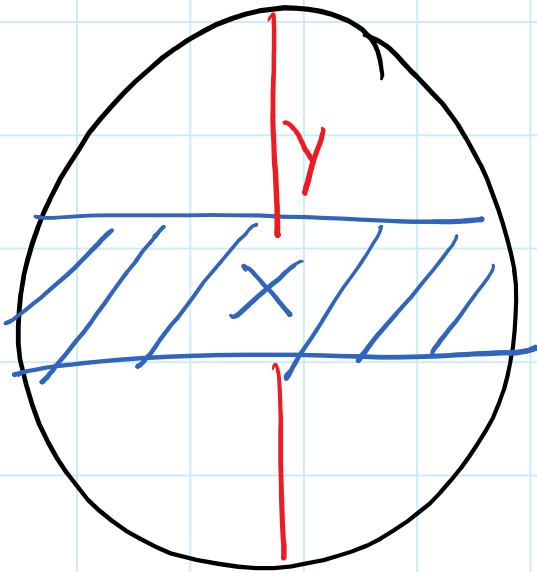
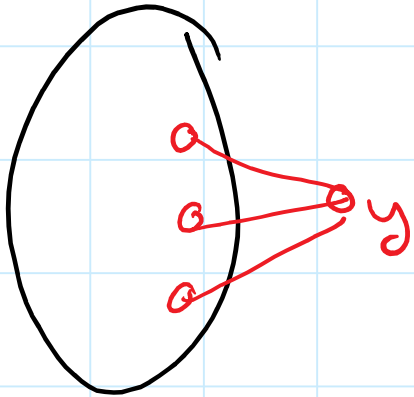
Can combine "unrelated" Naji solutions!



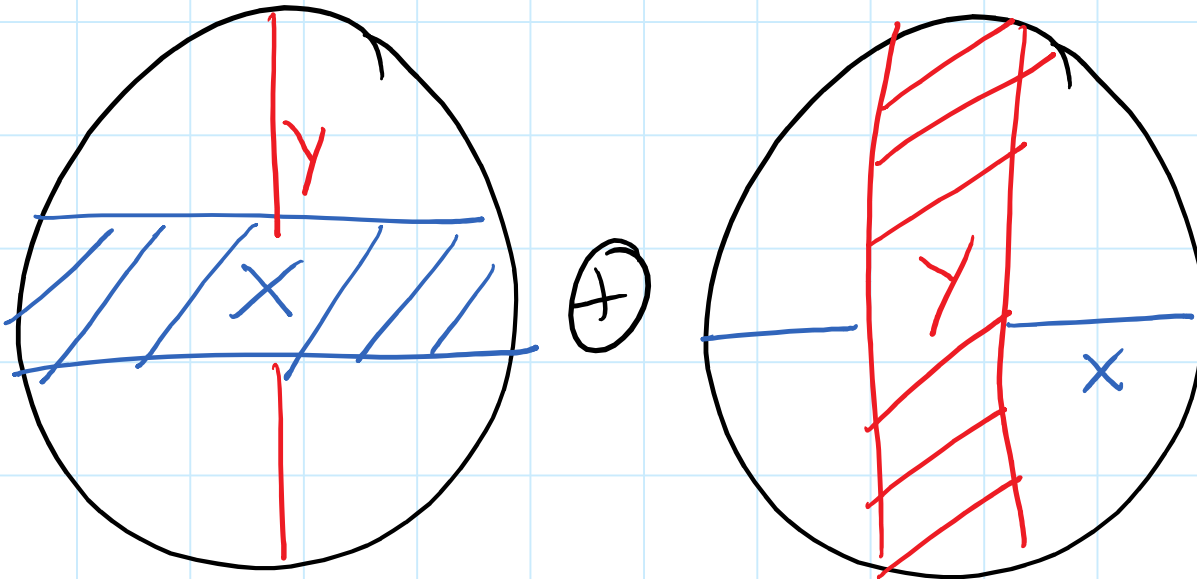
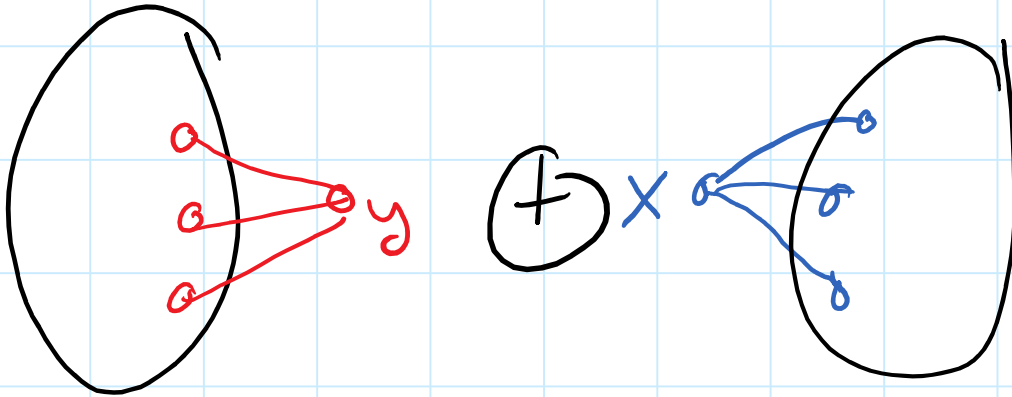
# Splits $\Rightarrow$ Inequivalent Diagrams



complete bipartite graph

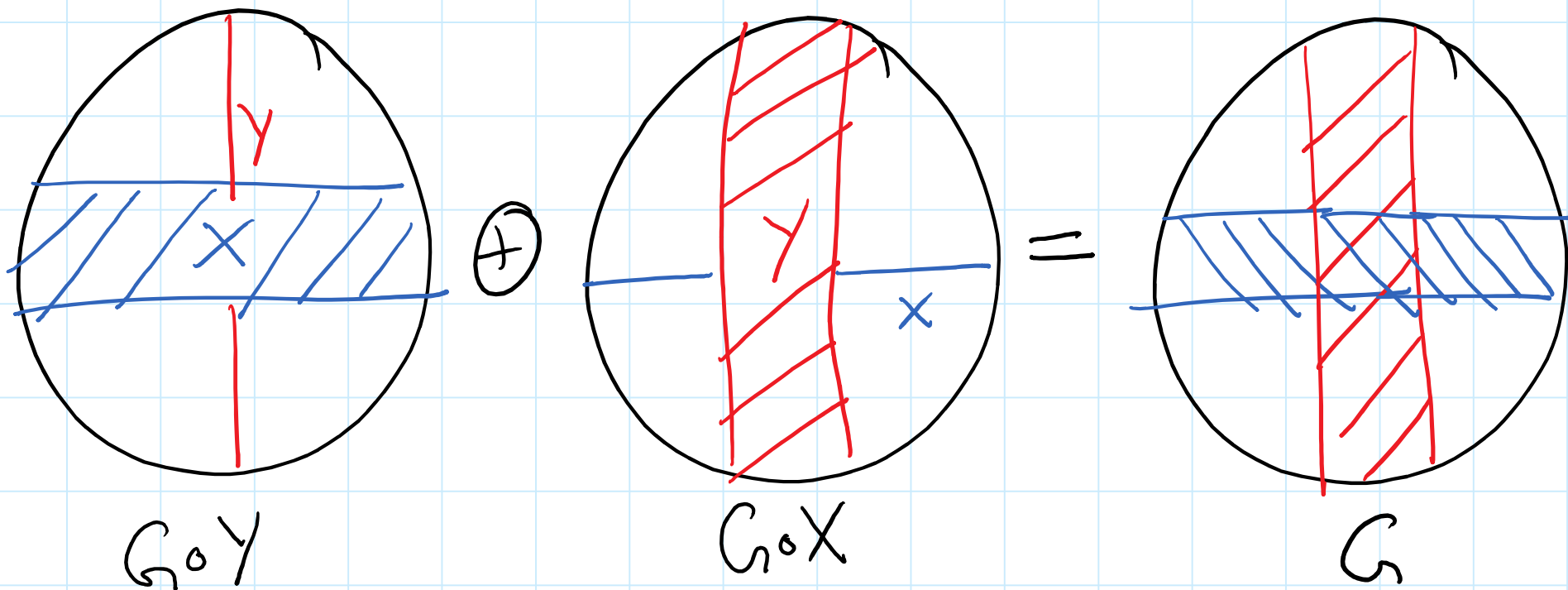
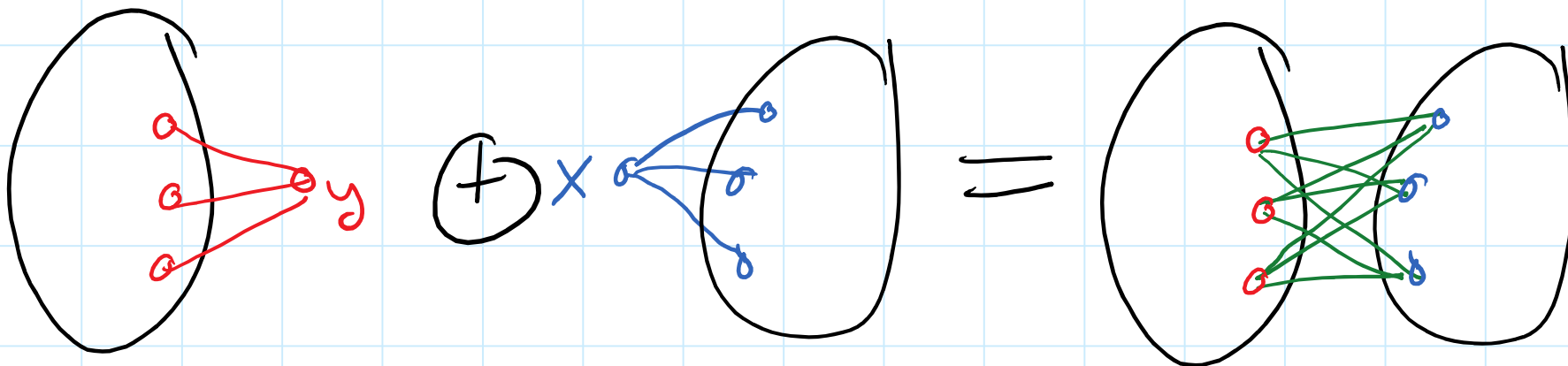


GoY



$G_0 Y$

$G_0 X$

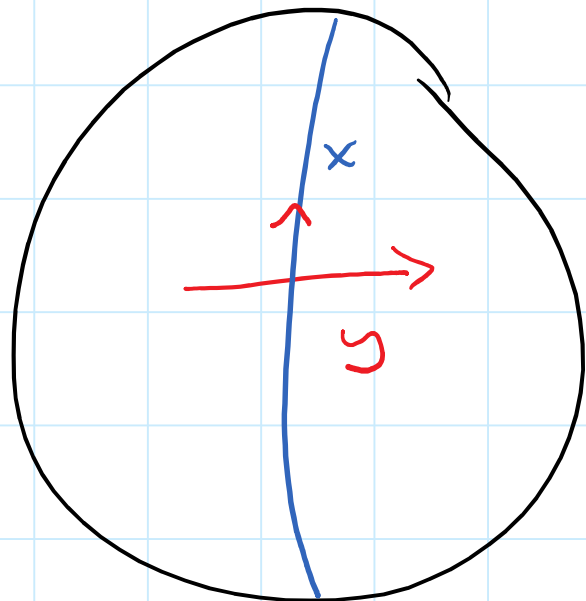


## Theorem (G,L)

Let  $\beta$  be a Naji solution for  $G$ . Then either:

(1):  $\beta = \beta(e)$

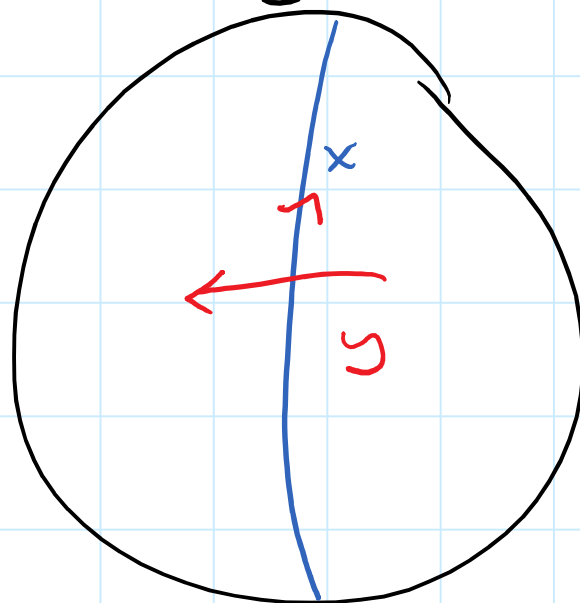
(2):  $G$  has a split.



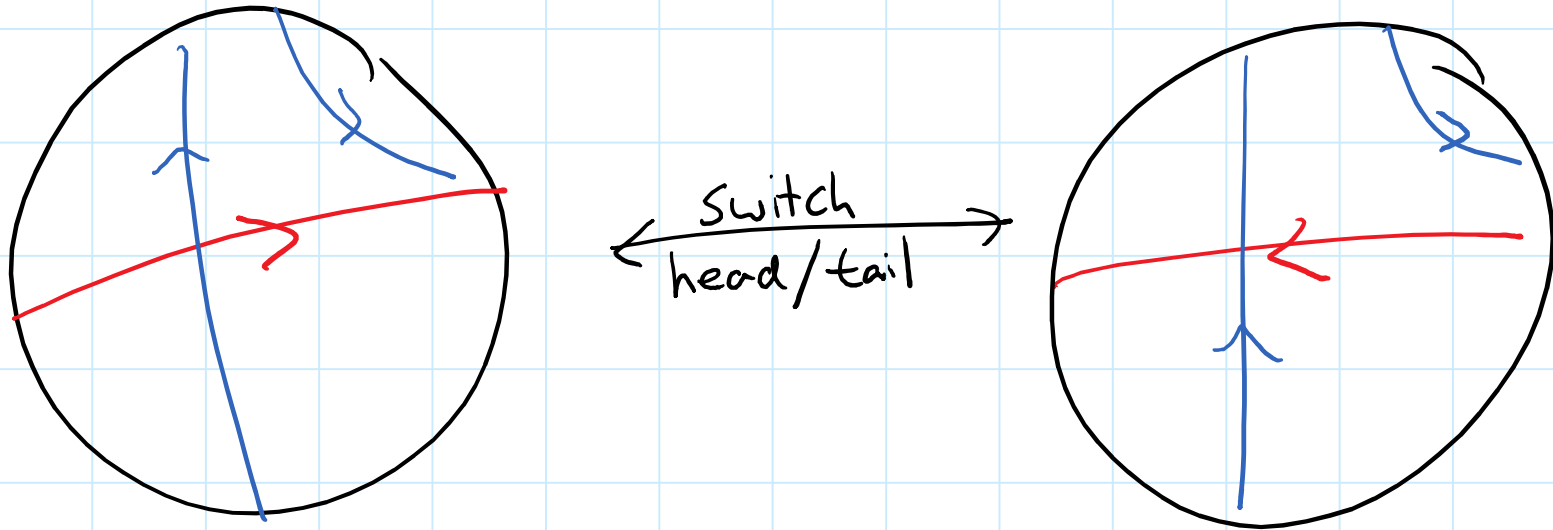
$$B(x,y) = 0$$

Each chord knows where the head of the next chord should go.

$$B(x,y) = 1$$



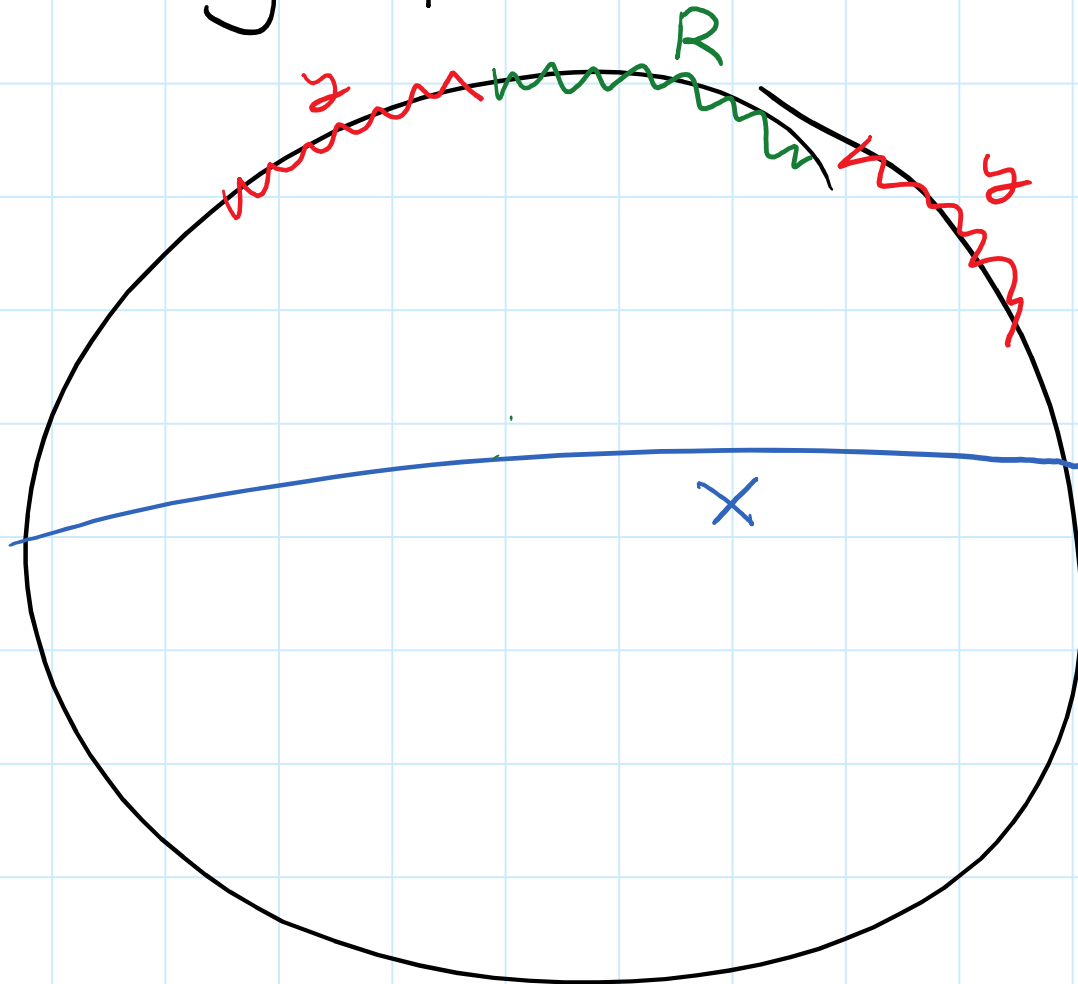
Moreover we can easily modify solution to find out where the tail should go. (Gasse)



$$\beta(e) = \int(v) + \beta(e')$$

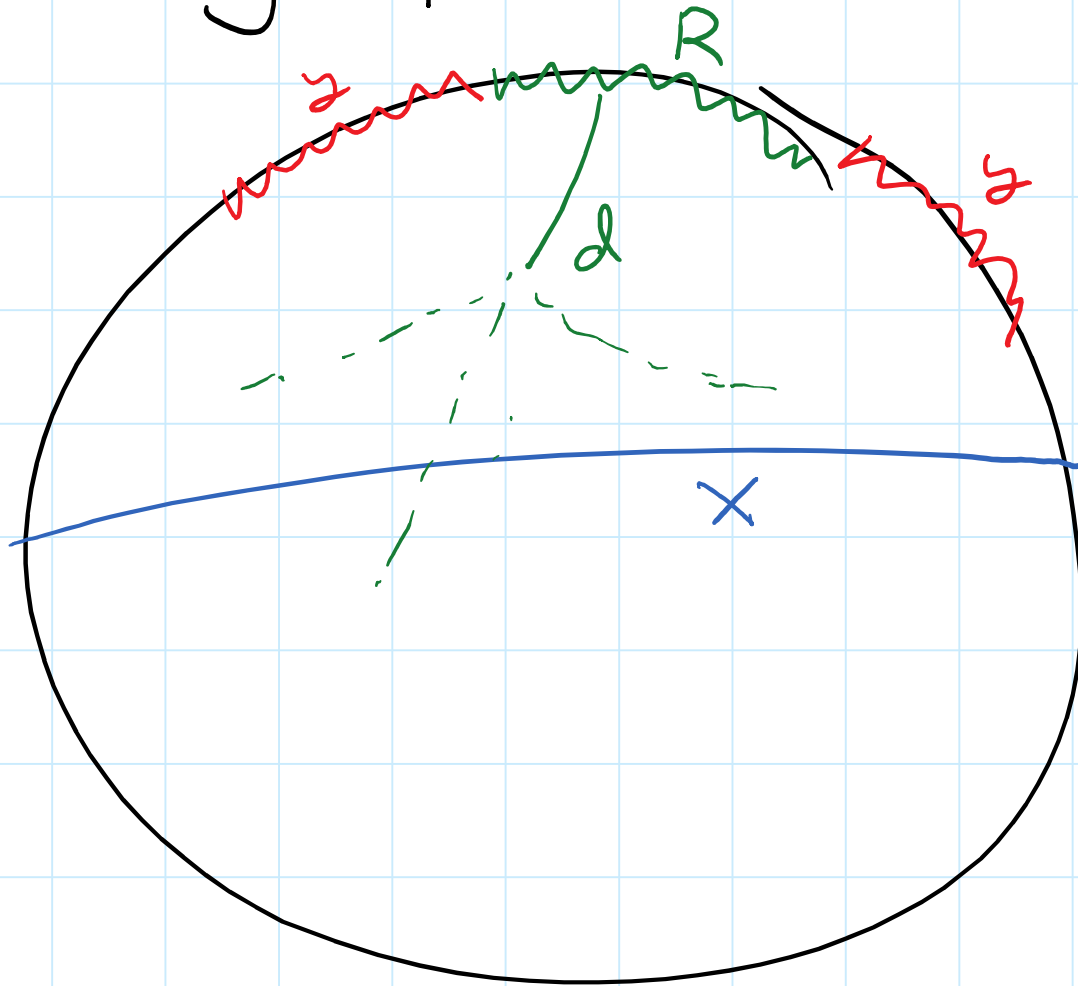
where  $\int_v(x,y) = 1$  if and only if  $x=v$   
or  $y=v$  and  $\{x,y\} \in E$ .

So long as we place the chords  
in a connected manner there is at  
most one way to place a new chord.

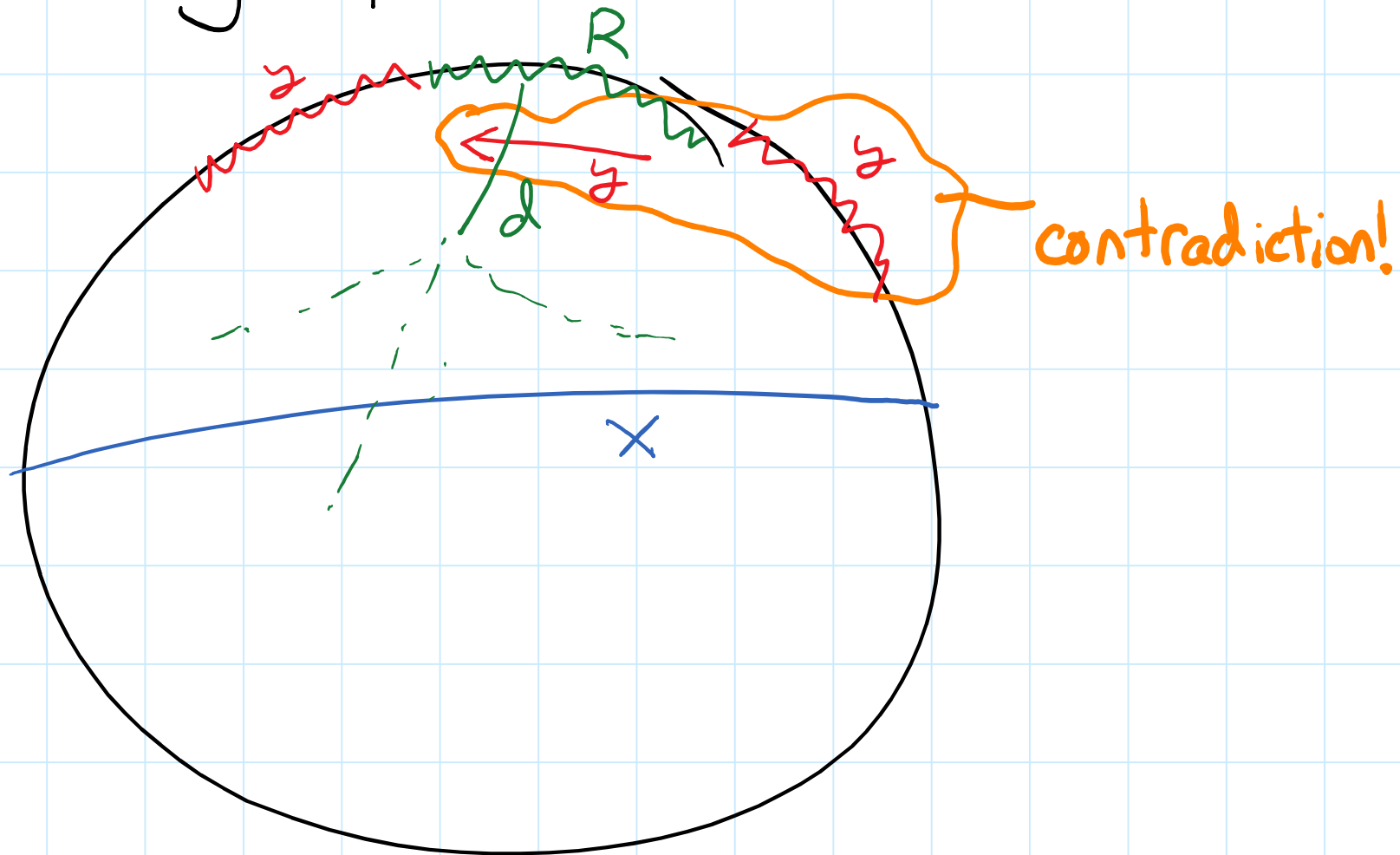




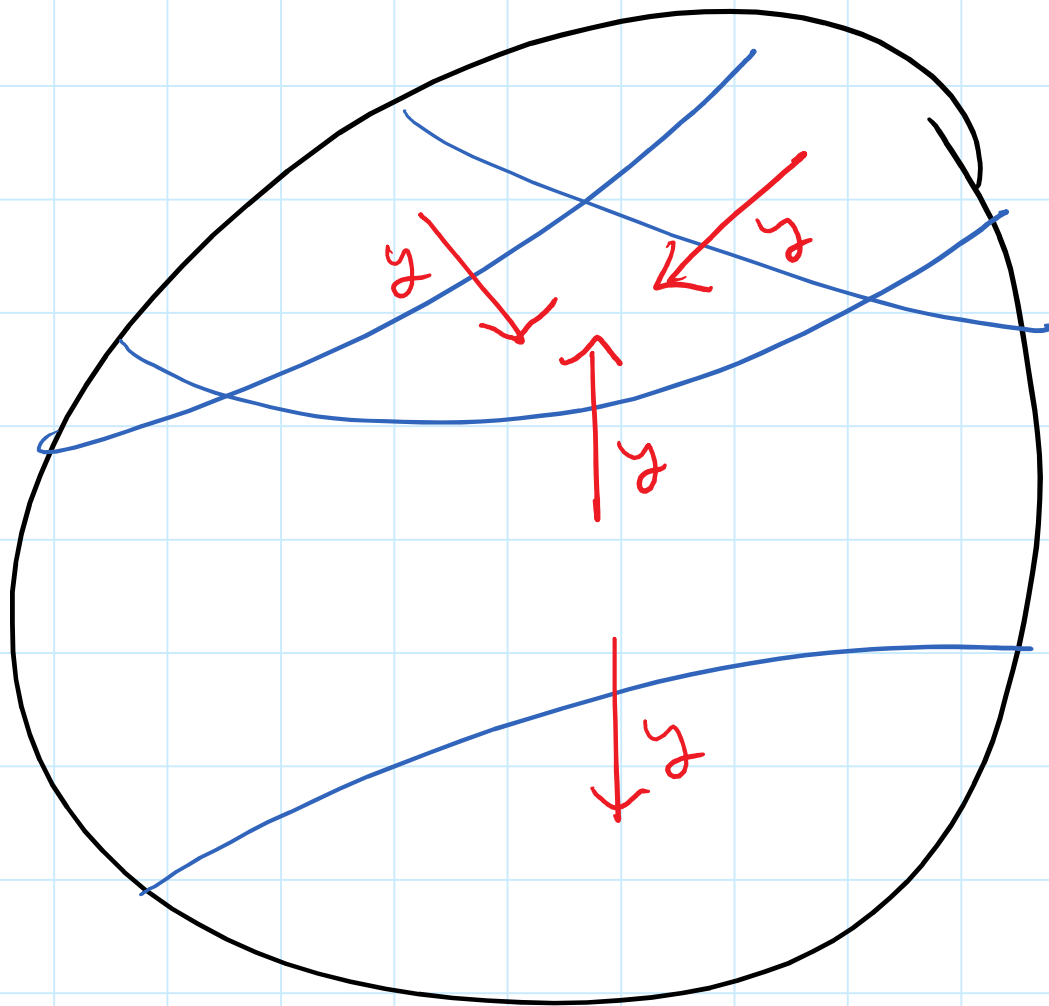
So long as we place the chords  
in a connected manner there is at  
most one way to place a new chord



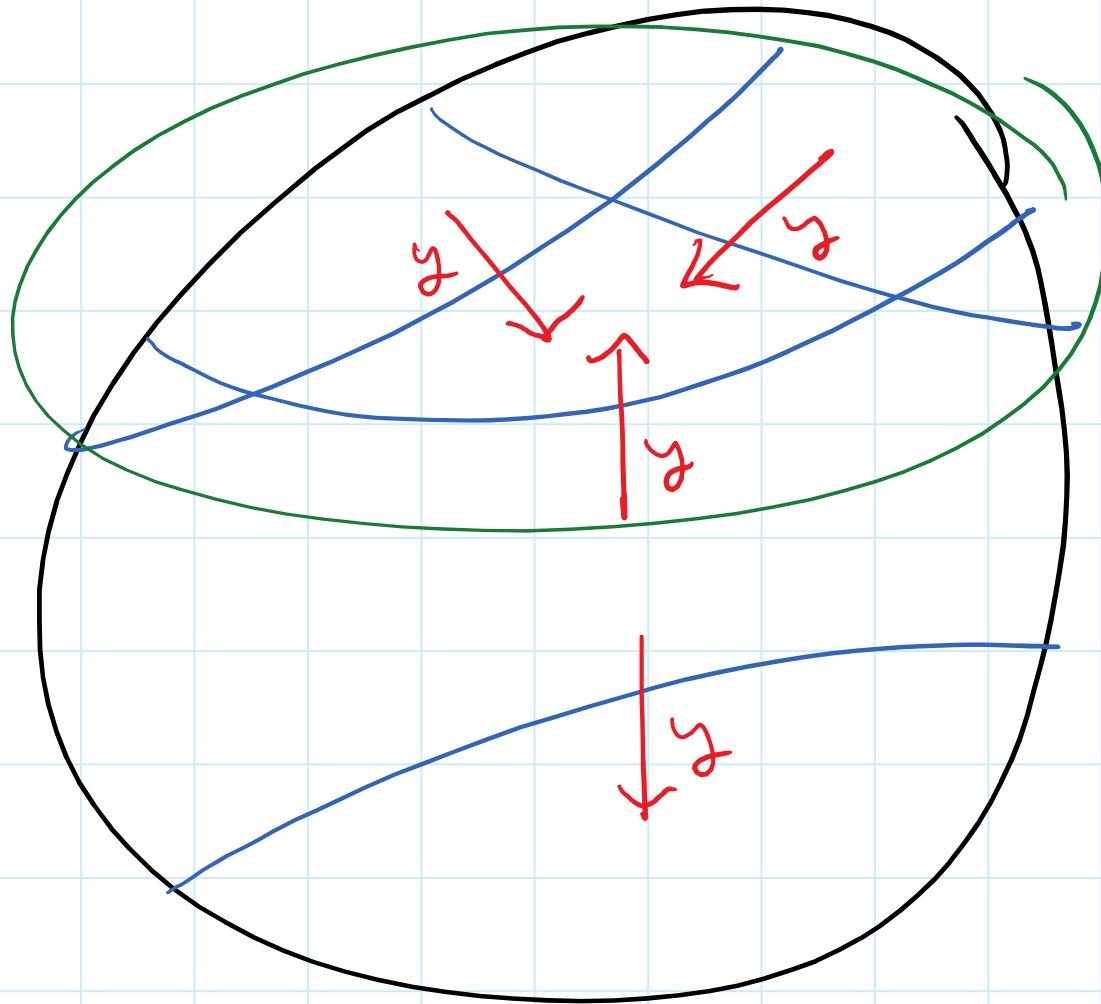
So long as we place the chords  
in a connected manner there is at  
most one way to place a new chord.



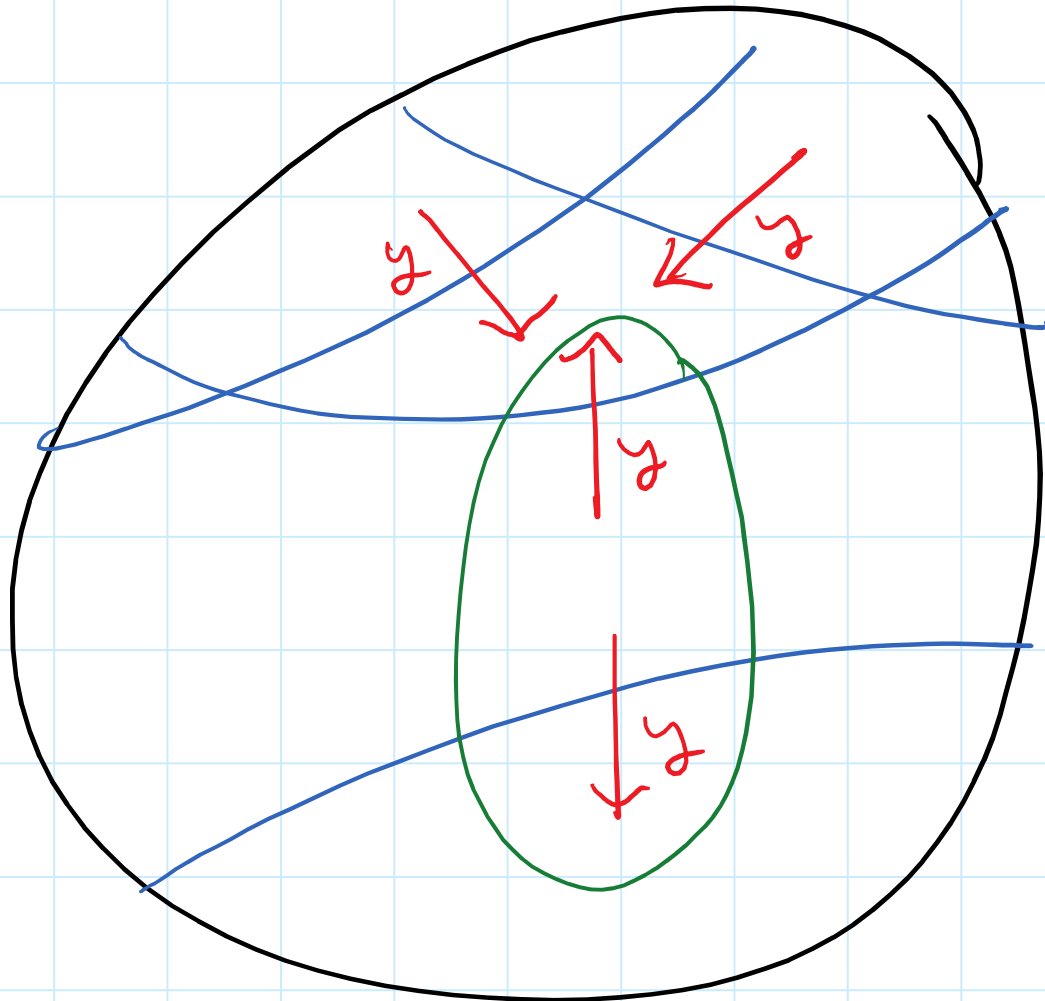
If something goes wrong:



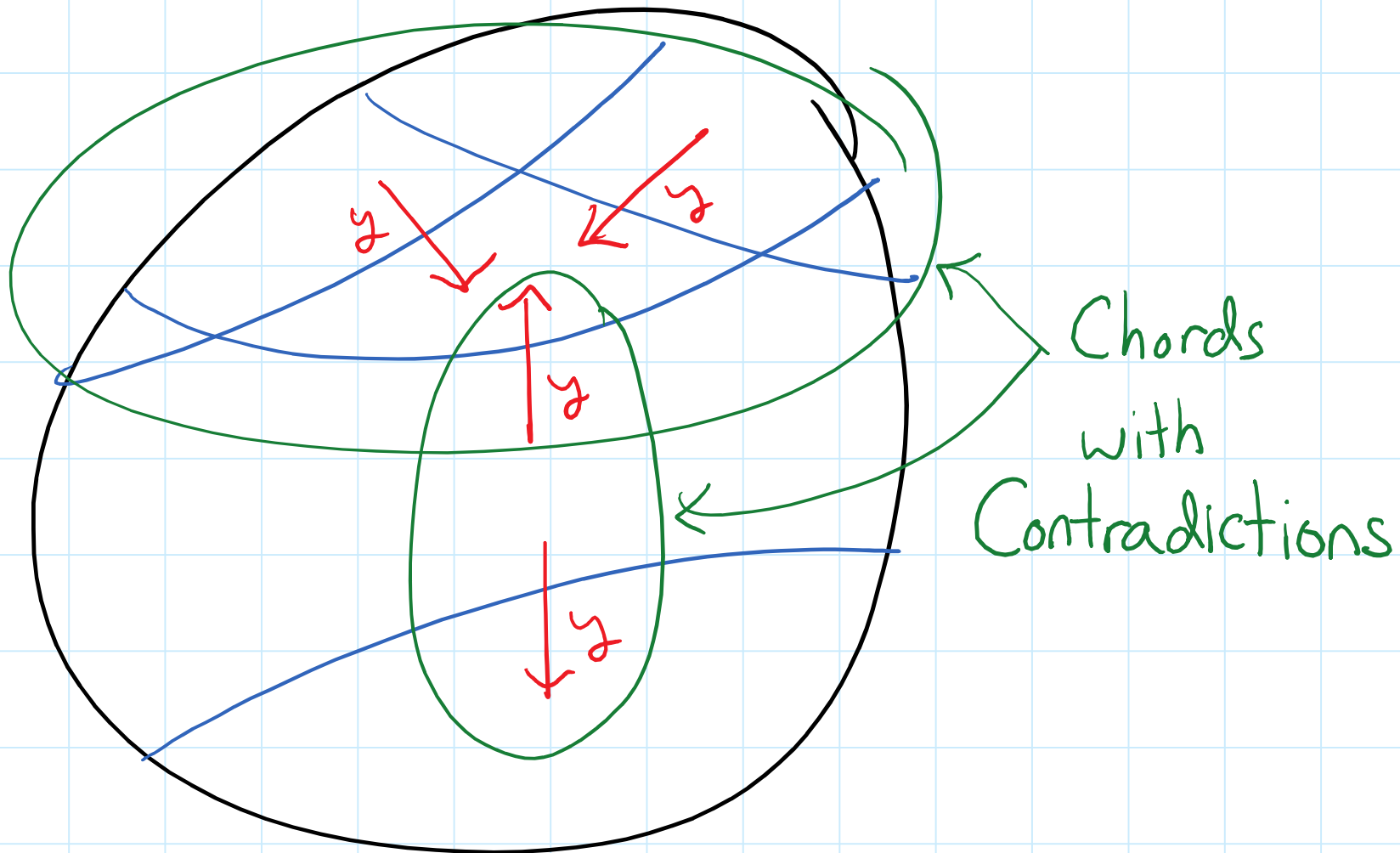
If something goes wrong:



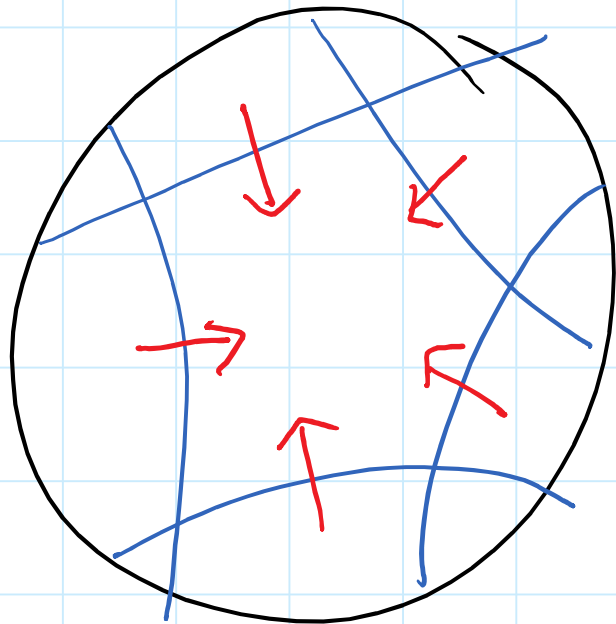
If something goes wrong:



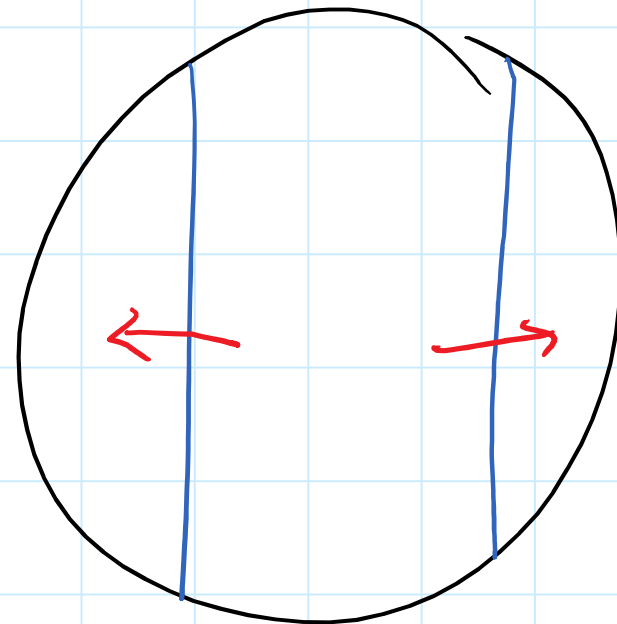
If something goes wrong:



Now if there are chords with contradictions  
we can find a minimal contradictory set:



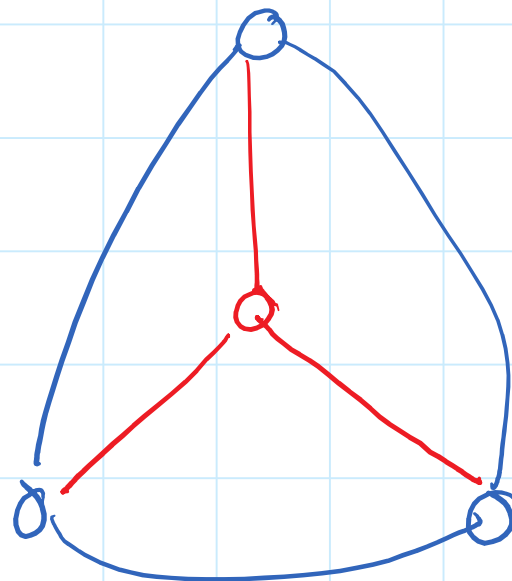
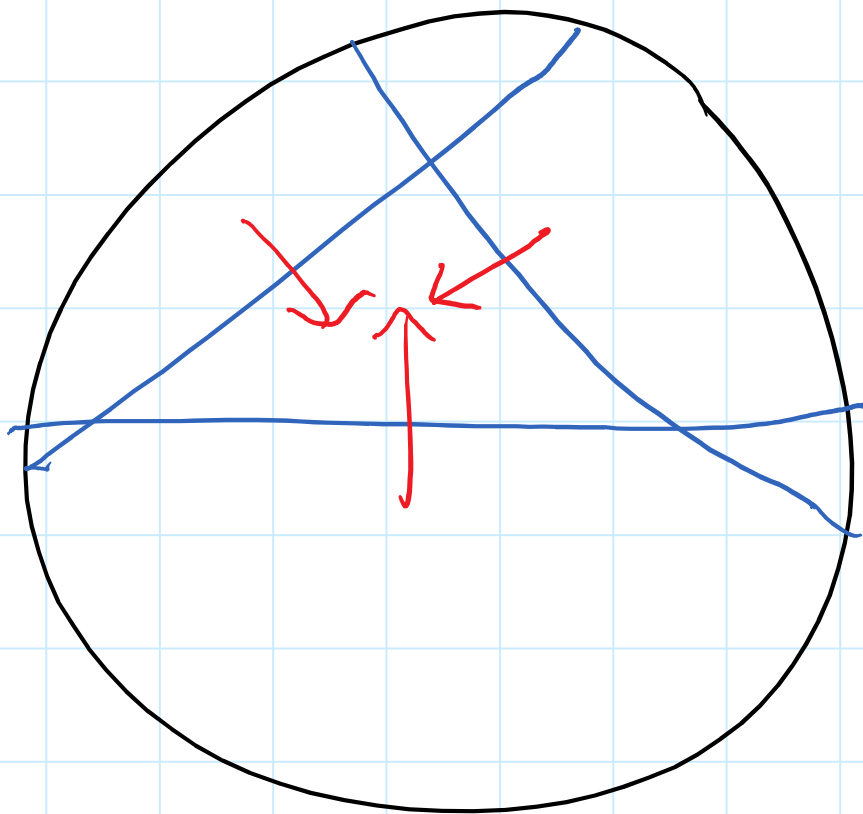
Induced  $C_k$ ,  
 $k \geq 3$



Two non-adjacent  
vertices

- If  $k \geq 5$  then  $G$  violates (N)

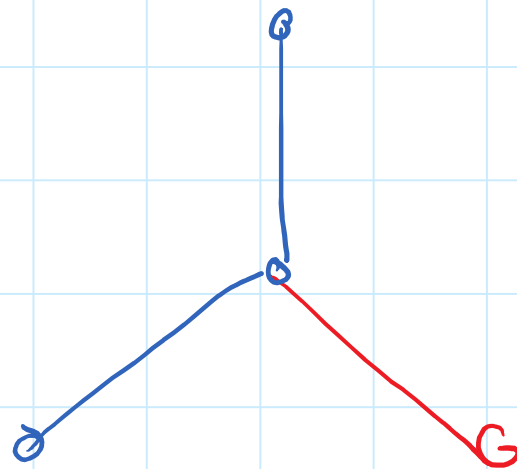
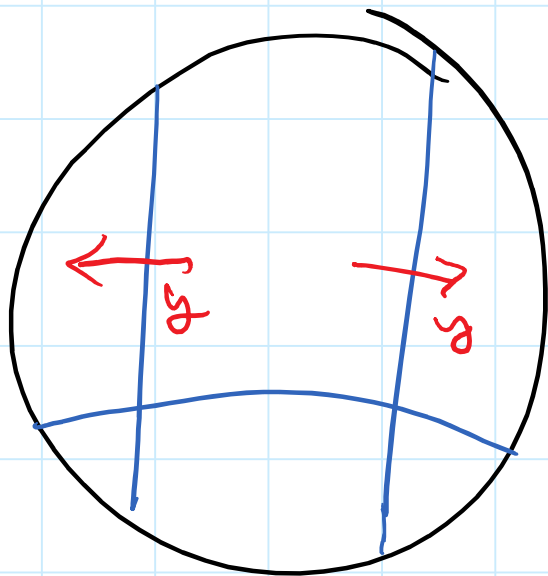
- So  $C_k \hookrightarrow C_3 + \text{axle} = K_4$

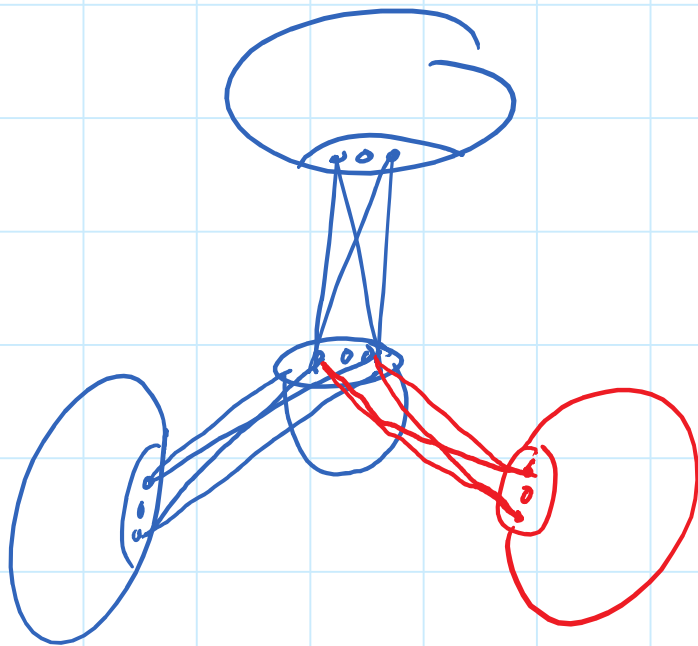
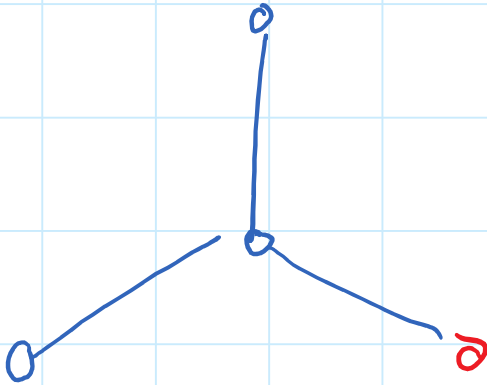


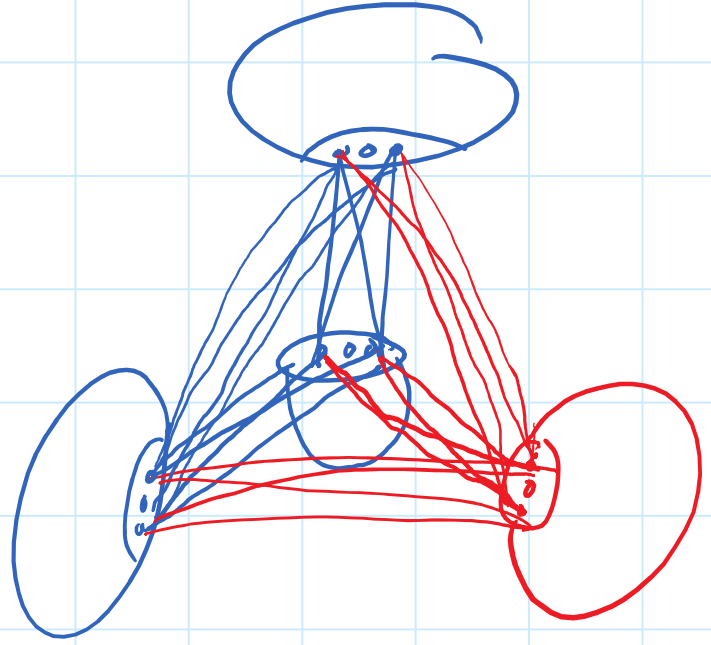
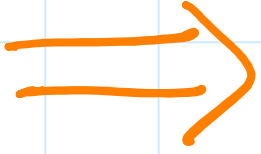
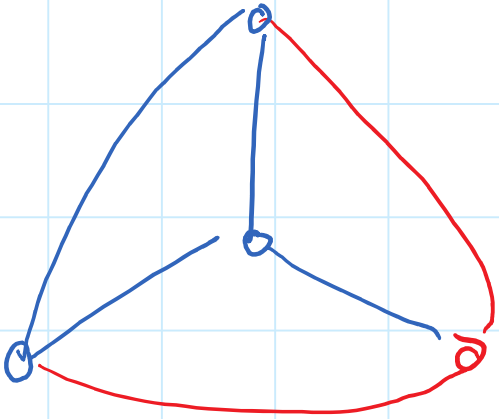
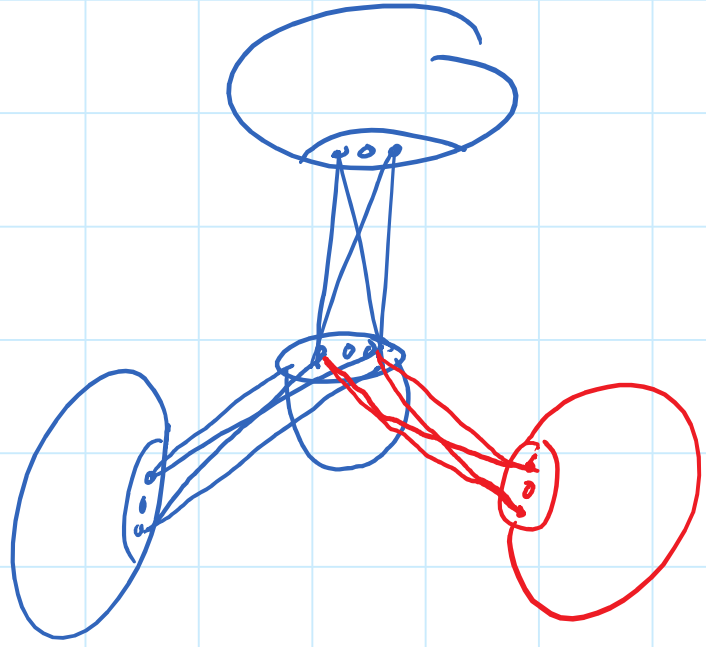
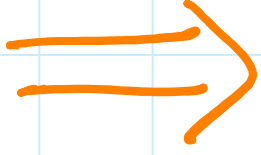
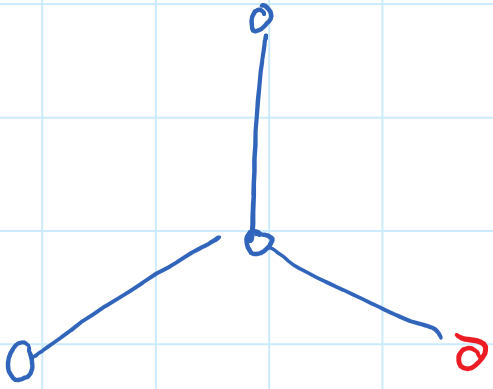


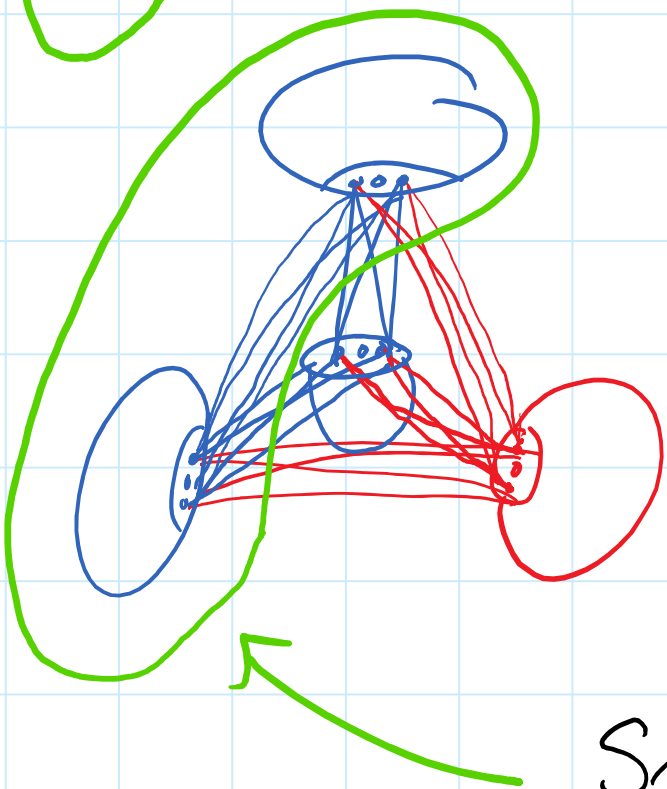
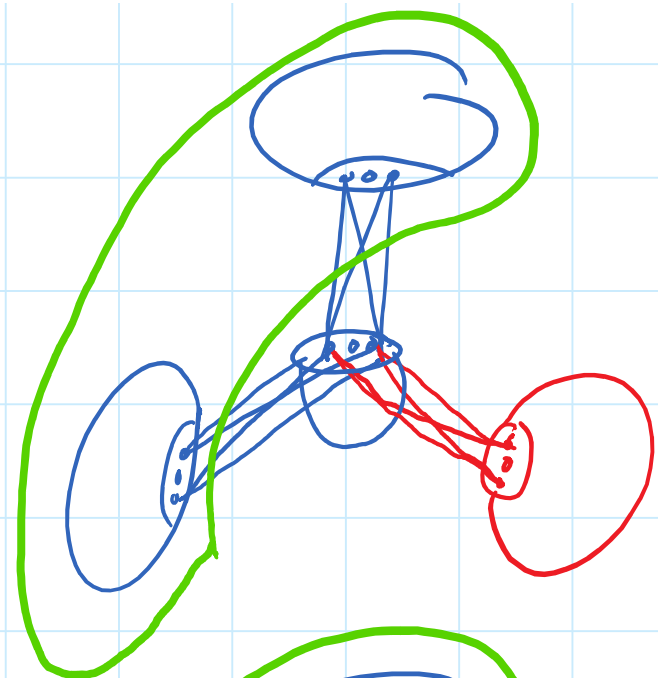
- As the chord diagram is connected  
there is a path between 2-non-adjacent  
vertices

- So one gets (by minimality):

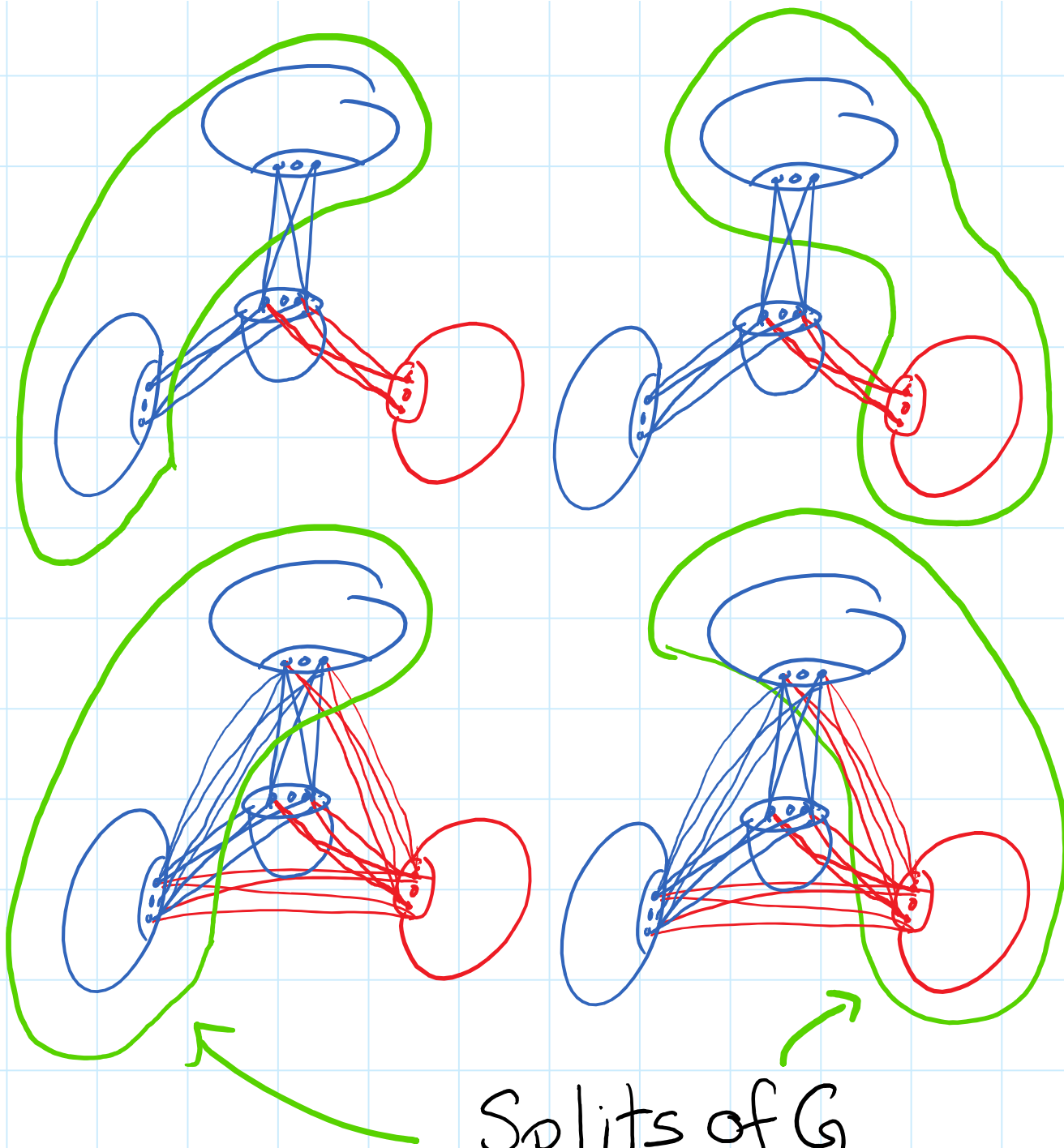




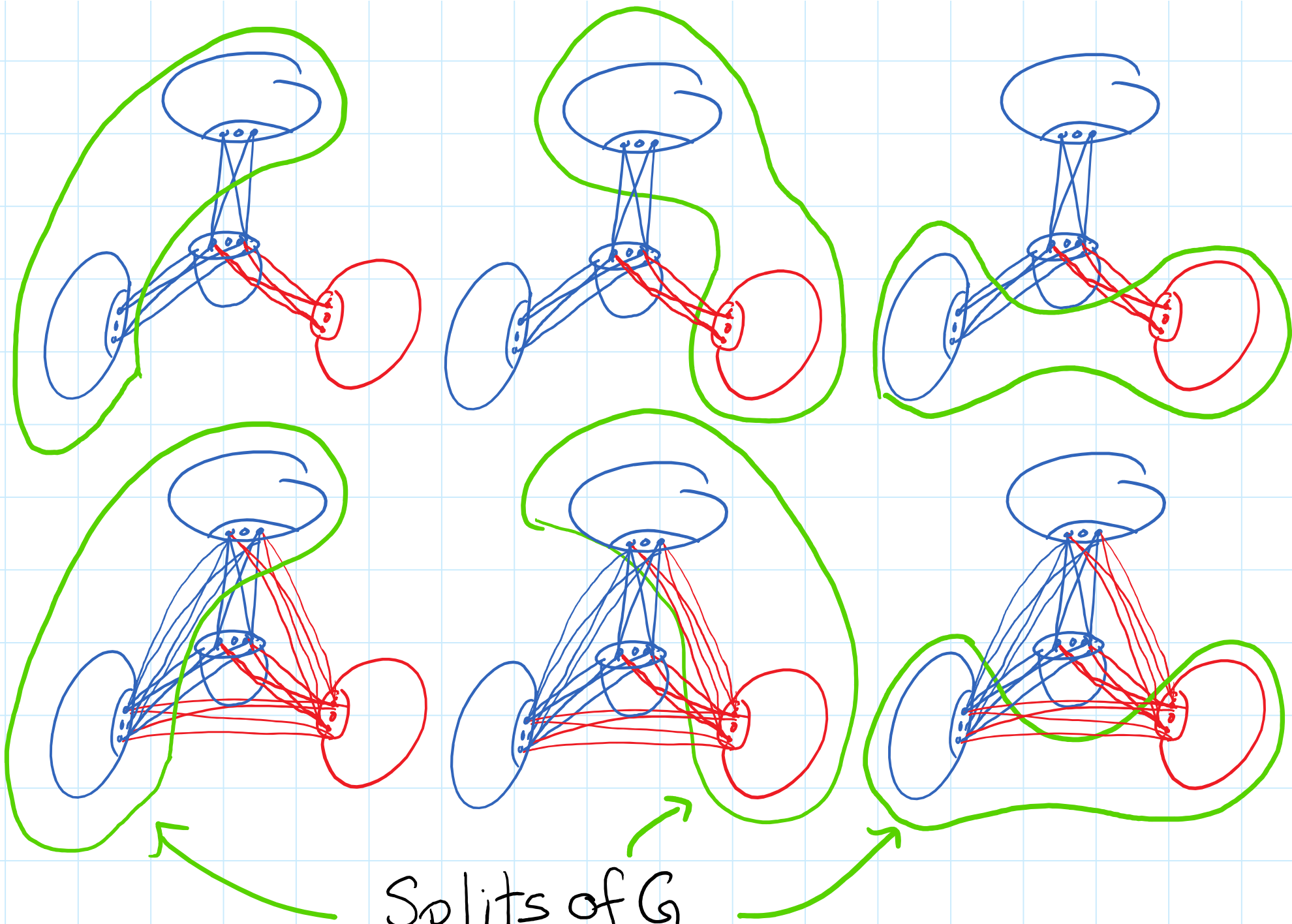




Splits of  $G$



Splits of  $G$



Splits of  $G$