

Lift & Project Systems Performing on the Partial Vertex Cover Polytope*

Konstantinos Georgiou and Edward Lee†

Department of Combinatorics and Optimization, University of Waterloo
200 University Ave. W, Waterloo ON, N2L 3G1 Canada
k2georgiou@uwaterloo.ca, e45lee@uwaterloo.ca

Abstract

We study integrality gap (IG) lower bounds on strong LP and SDP relaxations derived by the Sherali-Adams (SA), Lovász-Schrijver-SDP (LS_+), and Sherali-Adams-SDP (SA_+) lift-and-project (L&P) systems for the t -Partial-Vertex-Cover (t -PVC) problem, a variation of the classic Vertex-Cover problem in which only t edges need to be covered. t -PVC admits a 2-approximation using various algorithmic techniques, all relying on a natural LP relaxation. Starting from this LP relaxation, our main results assert that for every $\epsilon > 0$, level- $\Theta(n)$ LPs or SDPs derived by all known L&P systems that have been used for positive algorithmic results (but the Lasserre hierarchy) have IGs at least $(1 - \epsilon)n/t$, where n is the number of vertices of the input graph. Our lower bounds are nearly tight, in that level- n relaxations, even of the weakest systems, have integrality gap 1.

As lift-and-project systems have given the best algorithms known for numerous combinatorial optimization problems, our results show that restricted yet powerful models of computation derived by many L&P systems fail to witness c -approximate solutions to t -PVC for any constant c , and for $t = O(n)$. This is one of the very few known examples of an intractable combinatorial optimization problem for which LP-based algorithms induce a constant approximation ratio, still lift-and-project LP and SDP tightenings of the same LP have unbounded IGs.

As further motivation for our results, we show that the SDP that has given the best algorithm known for t -PVC has integrality gap n/t on instances that can be solved by the level-1 LP relaxation derived by the LS system. This constitutes another rare phenomenon where (even in specific instances) a static LP outperforms an SDP that has been used for the best approximation guarantee for the problem at hand.

Finally, we believe our results are of independent interest as they are among the very few known integrality gap lower bounds for LP and SDP 0-1 relaxations in which not all variables possess the same semantics in the underlying combinatorial optimization problem. Most importantly, one of our main contributions is that we make explicit of a new and simple methodology of constructing solutions to LP relaxations that almost trivially satisfy constraints derived by all SDP L&P systems known to be useful for algorithmic positive results (except the La system). The latter sheds some light as to why La tightenings seem strictly stronger than LS_+ or SA_+ tightenings.

1998 ACM Subject Classification G.1.6 Convex Programming, G.2.0 Combinatorial Algorithms

Keywords and phrases Partial vertex cover, combinatorial optimization, linear programming, semidefinite programming, lift and project systems, integrality gaps

Digital Object Identifier 10.4230/LIPIcs.FSTTCS.2014.199

* Research completed as part of the Undergraduate Research Program at the Department of Combinatorics and Optimization, University of Waterloo.

† Research supported by NSERC of Canada.



© Konstantinos Georgiou and Edward Lee;
licensed under Creative Commons License CC-BY

34th Int'l Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2014).
Editors: Venkatesh Raman and S. P. Suresh; pp. 199–211



Leibniz International Proceedings in Informatics

Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

1 Introduction

Let $G = (V, E)$ be a graph on n vertices and $t \in \mathbb{N}$, with $t \leq |E|$. A subset of vertices S that are incident to at least t many edges is called a *t-partial vertex cover*. In the *t-Partial-Vertex-Cover* (*t-PVC*) optimization problem, the goal is to find a *t-partial vertex cover* S of minimum size. *t-PVC* is a tractable optimization problem whenever $t = \Theta(1)$. In the other extreme, $|E|$ -PVC is exactly the classic **NP**-hard problem known as minimum Vertex-Cover (VC). As such, any hardness of approximation for VC translates to the same hardness for $|E|$ -PVC. In particular, $|E|$ -PVC is 1.36 and $(2 - o(1))$ hard to approximate assuming $\mathbf{P} \neq \mathbf{NP}$ [10] and the Unique Games Conjecture [18] respectively. Moreover, there exists an approximation preserving reduction from *t-PVC* to VC as long as $n/t = n^{\Theta(1)}$ [4]. Unlike VC, *t-PVC* is also known to be hard in bipartite graphs [5]. On the positive side, [16, 25, 31] have proposed 2-approximation algorithms even for the weighted version of *t-PVC* (see [20] for a wider family of results concerning partial covering problems). The common starting point of all these results is the standard 0-1 LP relaxation for *t-PVC* (see (*t-PVC-LP*) in Section 2.1). The best (asymptotic) approximation known for *t-PVC* relies on a SDP relaxation and achieves a $2 - \Omega(\log \log n / \log n)$ ratio [15].

A standard performance measure for convex-programming (LP or SDP) relaxations is the so-called integrality gap (IG), i.e. the worst possible ratio between the cost of the exact optimal solution and the cost of the relaxation. As a measure of complexity, IG upper or lower bounds are informative for two main reasons: (1) the majority of convex-programming based approximation algorithms attain an approximation ratio equal to the best provable upper bound on the IG. (2) Convex-programming relaxations can be seen as a restricted and static model of computation that can immediately witness using fractional solutions the existence of good integral and approximate solutions.

For a long series of combinatorial optimization problems, the best approximability known agrees with the IG of natural convex-programming relaxations. In contrast, all analyses for convex-programming relaxations for *t-PVC* [16, 25, 15] witness some integral solution with cost sol to the relaxation satisfying $sol \leq 2 \cdot rel + \Theta(1)$, where rel is the value of the relaxation. This leaves open the possibility that the IG of these relaxations is unbounded when the optimal solution has small enough cost. In fact, it was already known that the standard 0-1 relaxation (*t-PVC-LP*) has IG at least n/t . We establish the same IG for the SDP of [15].

However, the power of convex-programming for combinatorial optimization problems is not limited by the performance of the natural and static relaxations. A number of systematic procedures, known as lift-and-project (L&P) systems, have been proposed in order to reduce the IG of 0-1 LP relaxations $P \subseteq [0, 1]^m$ (the reader should think of P as the feasible region of a relaxation of some combinatorial problem). The seminal works of Lovász and Schrijver [23], Sherali and Adams [28], and Lasserre [21] give such systematic methods (LS, LS₊, SA, and La respectively).¹ Starting with the polytope P , each of the systems derives a sequence (hierarchy) of relaxations $P^{(r)}$ for $P \cap \{0, 1\}^m$ that are nested, preserve the integral solutions of P , and $P^{(m)}$ is exactly the integral hull of P (hence the IG of the last relaxation is 1 independently of the underlying objective). For these reasons, these systems are also known as hierarchies (of LP or SDP relaxations). More importantly, if P admits a (weak) separation oracle, then one can optimize a linear objective over the so-called level- r relaxation $P^{(r)}$ of

¹ LS₊ and SA systems derive stronger relaxations than the LS system, while LS₊, SA are incomparable. La derives SDPs that are at least as strong than relaxations derived by any other system.

all methods but the La system in time $m^{O(r)}$ (the same is true also for the La system if the initial relaxation has polysize). In other words, all L&P systems constitute “parameterized” models of computation for attacking intractable combinatorial optimization problems.

There are numerous combinatorial problems for which either L&P systems have given the best approximation algorithms known (with no matching combinatorial algorithms known), or with approximation guarantees matching the best combinatorial algorithms known. We refer the reader to [8] for a relatively recent survey.

For this reason, a long line of research has been devoted in proving IG lower bounds for relaxations derived by L&P systems, while any such result is understood as strong evidence of the true inapproximability of the combinatorial problem at hand. At the same time, an α IG for level- r relaxations derived by L&P systems implies that algorithms (for a restricted yet powerful model of computation) that run in time $m^{O(r)}$ cannot witness the existence of α -approximate solutions to the combinatorial problem. It is notable that examples of integrality gaps for L&P systems that are way off from the best approximability known for a combinatorial optimization problem are quite rare.

1.1 Our contributions and comparison to previous work

To the best of our knowledge, this is the first study of integrality gap lower bounds for lift-and-project tightenings of the natural 0-1 relaxation of t -PVC. Our starting point is the standard LP relaxation (t -PVC-LP) that has been used in all 2-approximation algorithms for weighted instances. We aim to derive strong integrality gap lower bounds for level- r relaxations derived by the LS_+ , SA and SA_+ systems, where r is as large as possible, and $t = O(n)$ (where n is the number of vertices in the input graph). It is worthwhile noticing that there is a number of very strong IG lower bounds known for VC in L&P systems, including IG of $2 - \epsilon$, for every $\epsilon > 0$, for level- $\Theta(n)$ LS LPs [27], level- $n^{\Theta(1)}$ SA LPs [6], level- $\Theta(\sqrt{\log/\log \log n})$ LS_+ SDPs [14], level-5 SA_+ SDPs [2], and IG of $7/6 - \epsilon$ and 1.36 for level- $\Theta(n)$ [26] and level- $n^{\Theta(1)}$ [32] La SDPs. Each of the aforementioned lower bounds imply directly the same IG lower bounds, for the same level relaxation and for the same system for (t -PVC-LP) by a straightforward reduction. Nevertheless, for the magnitude of t for t -PVC for which we establish our results (roughly speaking for $t \leq n/2$), and in which the problem makes the transition from tractable to intractable, our IG lower bounds are superconstant.

The majority of our results are negative. Our motivating observations are that (a) a simple graph instance is responsible for a n/t IG of the SDP of [15] (Proposition 2), on which the best algorithm know for t -PVC is based and (b) the level-1 LP derived by the LS system (which is strictly weaker than the LS_+ and SA systems) solves the same instances exactly (Proposition 5). This is a remarkable example of a simple LP that outperforms, even in a specific instance, an SDP that has been used for the best algorithm for a combinatorial problem (the authors are not aware of another similar example). It is natural then to ask whether relaxations derived by L&P systems can witness existence of 2-approximate solutions to t -PVC. We answer this question in the negative by proving strong IG lower bounds for all L&P systems (but the La system) that have been used for positive algorithmic results. For all these systems we show that as long as $n \geq 2r + 2t + 2$, the level- r relaxations have integrality gap at least $\binom{n-2r}{2}/t \cdot n$. As an immediate corollary, we see that the integrality gap of the starting LP (which is at least n/t) remains $(1 - \epsilon)\frac{n}{t}$ for level- $\Theta(n)$ LP and SDP relaxations. Our results could have also been stated as rank lower bounds of a certain knapsack-type inequality (the one certifying a good IG). Many similar results have appeared in the literature, e.g. [9, 22, 7], but they are all for polytopes that are of different structure than the partial vertex cover polytope.

The above negative results bring up another rare phenomenon; for the family of tractable combinatorial optimization problems t -PVC, for which $t = \Theta(1)$, L&P-relaxations have unbounded discrepancy. The authors are aware only of one more similar result [24]. This is in contrast to many combinatorial optimization problems, and in particular VC, for which constant-level L&P-relaxations either have integrality gaps matching the best approximability or they even solve tractable variations of the problems. Finally, due to the approximation preserving reduction from VC to t -PVC [4], when $t = n^{\Theta(1)}$, our results also imply that L&P systems applied on the t -PVC standard polytope cannot yield new insights for the NP-hardness inapproximability of VC.

We believe that our results are of independent interest also for two more reasons. The first reason is that relaxation (t -PVC-LP), for which we establish strong IG L&P lower bounds, is defined over two types of variables, i.e. vertex and edge variables corresponding to different semantics. IG lower bounds for L&P relaxations of such polytopes are very rare (the authors are aware only of one such result [19]). The second reason is that it is not well understood under which conditions semidefinite programming delivers better algorithmic properties than linear programming. Especially for LPs, the probabilistic interpretation of SA system (deriving the strongest LPs known), on which we elaborate below, has unified our understanding both for positive and negative results. When it comes to SDPs, one needs to employ seemingly stronger arguments that enhances the probabilistic interpretation of the systems with a geometric substance. Interestingly, with our technique for showing L&P lower bounds, we make explicit that it is possible to devise solutions to LP relaxations that satisfy many PSD conditions, almost trivially. For this we identify a generic and remarkably simple condition of solutions to LP relaxations that can fool a large family of PSD constraints (for a high level explanation of the condition see Section 1.2). We hope that this simple observation can help towards bridging our understanding for LP and SDP relaxations.

1.2 Our techniques

For our main results we employ some standard and generic techniques for constructing vector solutions for convex relaxations derived by the SA system. Then we identify a condition special to our solution that allows us to argue that the same construction is robust against SDP tightenings. Our IG instance is the unweighted clique on n vertices, which for all t , admits an optimal solution of cost 1. This IG construction suffers a decay that is proportional to $\binom{n-2r}{2}$. The decay with r is unavoidable, at a high level, due to that level- r relaxations solve accurately local subinstances induced by r many elements corresponding to variables. Since our LP relaxation has edge variables, the removal of r many edges induces a clique of $n - 2r$ vertices. Since we still have $\binom{n-2r}{2}$ edges, each edge needs to be covered “on average” $t/\binom{n-2r}{2}$ fractional times. Due to the symmetry imposed in our solutions, this is also the contribution of each vertex in the objective.

Establishing the SA IG lower bound: A common and generic approach for constructing SA solutions is to use the probabilistic interpretation of the system, first introduced in [17], and that is implicit in all our arguments of Section 3. At a high level, the curse and the blessing of the SA system is that level- r solutions are convex combinations of (LP feasible) vectors that are integral in any set of r many variable-indices. These convex combinations can be interpreted as families of distributions of *feasible* integral solutions for subsets of the input instance of size- r (hence subsets of variables as well), that additionally enjoy the so-called *local-consistency* property: distributions over different subinstances should agree on the solutions of the common sub-subinstance. Designing such probability distributions

over sets of indices that also enclose the support of any constraint gives automatically a solution to the level- r SA. Finding however such distributions is in general highly non trivial, especially when aiming for a big integrality gap.

The previous recipe is not directly applicable to the t -PVC polytope, as it has a defining facet that involves all edges of the input graph. This means that had we blindly tried to find families of probability distributions as described above, then we would have unavoidably defined distributions of feasible solutions in the integral hull. Our strategy is to deviate from the generic probabilistic approach, and focus first on satisfying constraints of the SA relaxation of relatively small support.

At a high level, the novelty of our approach is that we do not explicitly define locally consistent distributions of local 0-1 assignments, one for each subset of variables of bounded size, rather we achieve this implicitly. One of the advantages of our construction is that it is surprisingly simple. Specifically, we define a *global* distribution of 0-1 assignments as follows: each of the vertices is chosen in the solution independently at random, and with negligible probability, and covered edges are those incident to at least one chosen vertex.

The locally consistent distributions, that we need to associate each subset of variables A with, are obtained by restricting the global distribution onto the subinstance induced by A . This trick can be thought as a vast generalization of the so-called correction-phase (or expansion recovery) that is common to all SA lower bounds, although it is sometimes hidden in the technicalities of the proofs ([13] is a good example where the correction phase is made explicit). According to this trick, set A is effectively blown up (or “corrected”) to a big enough superset \bar{A} with certain structural properties. This allows for sampling almost uniformly at random over local 0-1 assignments (of variables in \bar{A}) that can be easily seen to induce consistent local distributions, whereas the same task seems to be impossible to be realized directly on A . Interestingly, \bar{A} is the whole instance in our case.

Our global distribution has a special property that it always satisfies all linear constraints of the t -PVC polytope but the one demand-constraint, i.e. the constraint that requires t many edges to be covered. In particular, the proposed vector solution is a convex combination of exponentially many solutions in the integral hull and of the outlier all-0 vector. In fact our global distribution assigns probability $1 - o(1)$ to the latter vector, which is also responsible for the large integrality gap.

Notably, there is no generic reason to believe that such a vector solution satisfies the almost global constraint of the t -PVC polytope that involves all edges. To that end, we take advantage of the fact that we do *not* need to define feasible solutions of the whole instance in every small subinstance. This means that if presented with a small subinstance of the input graph, we are allowed in principle to cover zero edges in that subgraph with positive probability, as long as we do cover t many edges in the complement. That said, constraints of large support cannot be treated probabilistically with respect to the global distribution. Instead, we deal with such constraints almost algebraically (in contrast to the majority of SA constructions), as one would normally do for a standard LP. More specifically, we rely on the fact that when we condition on covering zero edges in a subclique of size at most $2r$, edges that do not touch this subclique are covered independently at random with significant probability compared to how many edges are left. Linearity of expectation then can prove for us that the demand constraint is indeed satisfied.

Establishing IG lower bounds for SDP hierarchies: Showing that our SA vector solution is robust against SDP tightenings is by construction very easy. The reason is that all SDP hierarchies (that have been used for positive algorithmic results), except the La system,

distinguish constraints between those imposed by the starting 0-1 relaxation, and that are *always* linear, and PSD constraints that are valid for *all* 0-1 assignments (independently of the starting relaxation). As a result, any IG lower bound for strong LP relaxations that is based on a solution that comes from a global distribution of 0-1 assignments immediately translates into the same IG for a series of SDP hierarchies. A natural question that is raised is whether such global distributions of 0-1 assignments can be used to fool strong LP relaxations (and we answer this in the positive as we explain above). The second question that we raise is whether our solution is robust also against Lasserre tightenings. We answer this in the negative in Section 4.2.

2 Preliminaries

We denote by $\mathbf{1}_n$ the all-1 vector of dimension n , and we drop the subscript, whenever the dimension is clear from the context. Similarly, by all- α vector we mean the vector $\alpha\mathbf{1}$. For a fixed set of indices $[m] := \{1, \dots, m\}$, we denote by \mathcal{P}_r all subsets of $[m]$ of size at most r (for the partial vertex cover polytope and for a graph $G = (V, E)$, we will use $[m] = V \cup E$). For some $y \in \mathbb{R}^{\mathcal{P}_{r+1}}$, we denote by \mathcal{Y} the so-called *moment matrix* of y that is indexed by \mathcal{P}_1 in the rows and by \mathcal{P}_r in the columns, with $\mathcal{Y}_{A,B} = y_{A \cup B}$. In other words, $\mathcal{Y} \in \mathbb{R}^{|\mathcal{P}_1| \times |\mathcal{P}_r|}$ whenever $y \in \mathbb{R}^{\mathcal{P}_{r+1}}$, whereas \mathcal{Y} is a square symmetric matrix if $r = 1$. Finally, we denote by $\{\mathbf{e}_I\}_{I \in \mathcal{P}_r}$ the standard orthonormal basis of \mathcal{P}_r , so that $\mathcal{Y}\mathbf{e}_A$ is the column of \mathcal{Y} indexed by set A .

Note: In the interest of space, we have omitted some proofs from this version of the paper; we refer the reader to [12] for a full presentation.

2.1 Problem Definition and a Natural LP Relaxation

Given an integer t , and a graph $G = (V, E)$ with vertex weights $w_i \in \mathbb{R}_+$ for each $i \in V$, t -PVC can be alternatively defined as the following optimization problem where variables $\{x_q\}_{q \in V \cup E}$ are further restricted to be integral.

$$\min \sum_{i \in V} w_i x_i \quad (t\text{-PVC-LP})$$

$$\text{s.t. } x_i + x_j \geq x_e, \quad \forall e = \{i, j\} \in E \quad (1)$$

$$\sum_{e \in E} x_e \geq t \quad (2)$$

$$0 \leq x_q \leq 1 \quad \forall q \in V \cup E \quad (3)$$

Below we focus on uniform instances, in which $w_i = 1$, for all $i \in V$. We denote the set of feasible solutions of the above LP as $P_t(G)$, or much simpler as P_t when the underlying graph is clear from the context, and we call it the t -partial vertex-cover polytope. For each edge e , the reader should understand x_e as the 0-1 indicator variable that says whether e will be among the (at least) t many that will be covered by some vertex, while for each vertex i , the 0-1 variables x_i indicate whether vertex i is chosen in the solution.

(t -PVC-LP) is the starting point for the 2-approximation algorithm for t -PVC in [4], and a $2 - \Theta(1/d)$ approximation for unweighted instances, where d the maximum degree of the input graph, in [29, 11]. Strictly speaking, the analysis that guarantees the 2-approximability is not relative to the performance of the LP for all instances, as in fact (t -PVC-LP) has an unbounded integrality gap.

► **Observation 1** (Star-graph fools (t -PVC-LP) [25]). Consider the unweighted star-graph $G = (V, E)$ with $V = 1, \dots, n, n+1$, and edges $\{n+1, i\} \in E$, for $i = 1, \dots, n$. The optimal solution to t -PVC is 1, for every $t \in \mathbb{N}$. In contrast, consider the feasible solution to (t -PVC-LP) that sets $x_e = x_{n+1} = t/n$ for all $e \in E$, and the rest of variables equal to 0. This gives a solution of cost t/n , hence the integrality gap of (t -PVC-LP) is at least n/t .

This is also true for the SDP relaxation of t -PVC that gives the best known approximation algorithm.

► **Proposition 2.** For all $t \leq n/2$, the SDP of [15] has integrality gap at least n/t when the input is the star-graph of Observation 1.

2.2 Hierarchies of LP and SDP relaxations

In this section we introduce families of LPs and SDPs derived by the so-called LS, LS₊ [23] and SA [28] systems. Starting with a polytope $P \subseteq [0, 1]^m$, each of the LS₊ and SA systems derives a nested sequence of relaxations $\{P^{(r)}\}_{r=1, \dots, m}$, such that $P^{(m)} = \text{conv}(P \cap \{0, 1\}^m)$, while under mild assumptions one can optimize over $P^{(r)}$ in time $m^{O(r)}$. For an instance $G = (V, E)$ of t -PVC, our intention is to derive and study this sequence of relaxations starting with $P = P_t(G)$, i.e. the feasible region of the standard LP relaxation (t -PVC-LP), hence setting $|m| = |V| + |E|$. For the sake of simplicity, we adopt a unified exposition of the systems (see [22] for a more abstract exposition of lift-and-project systems).

For technical reasons, it is convenient to apply a standard homogenization to polytope P as follows: variables x_p are replaced by $\bar{x}_{\{p\}}$ and each constraint $a^T x \geq b$ is replaced by $a^T \bar{x} \geq b \bar{x}_\emptyset$. Adding the constraint $\bar{x}_\emptyset \geq 0$ along with the previous constraints define a cone that we denote by K . Clearly $K \cap \{\bar{x}_\emptyset = 1\}$ is exactly polytope P . Next we define a sequence of SDP refinements of an arbitrary 0-1 polytope, proposed by Lovász and Schrijver [23], and that is commonly known in the literature as the LS₊-hierarchy (of SDPs).

► **Definition 3** (The LS₊ system). Let $K^{(0)} := K$ be a conified polytope $P \subseteq [0, 1]^m$. The level- r LS₊-tightening of $K^{(0)}$ is defined as the cone

$$K^{(r)} = \left\{ x \in \mathbb{R}^{\mathcal{P}_1} : \exists y \in \mathbb{R}^{\mathcal{P}_2} : \begin{array}{l} \mathcal{Y} \succeq \mathbf{0}, \quad \mathcal{Y} \mathbf{e}_\emptyset = x \text{ and} \\ \forall i \in [m], \quad \mathcal{Y} \mathbf{e}_{\{i\}}, \mathcal{Y} (\mathbf{e}_\emptyset - \mathbf{e}_{\{i\}}) \in K^{(r-1)} \end{array} \right\}$$

The level- r LS₊ refinements (tightenings) $\mathcal{N}_+^{(r)}(P)$ of P is obtained by projecting $K_+^{(r)}$ onto $x_\emptyset = 1$, i.e. $\mathcal{N}_+^{(r)}(P) = K_+^{(r)} \cap \{x \in \mathbb{R}^{\mathcal{P}_1} : x_\emptyset = 1\}$.

Next we introduce the SA system defined by Sherali and Adams [28], and that derives a sequence of LP relaxations (and not SDP relaxations).

► **Definition 4** (The SA system). Let K be a conified polytope $P \subseteq [0, 1]^m$. The level- r SA tightening of K is defined as the cone

$$M^{(r)} = \left\{ x \in \mathbb{R}^{\mathcal{P}_1} : \exists y \in \mathbb{R}^{\mathcal{P}_{r+1}} : \begin{array}{l} \mathcal{Y} \mathbf{e}_\emptyset = x, \text{ and} \\ \forall Y, N \text{ with } Y \cup N \in \mathcal{P}_r, \quad \mathcal{Y} \sum_{\emptyset \subseteq T \subseteq N} (-1)^{|T|} \mathbf{e}_{Y \cup T} \in K \end{array} \right\}$$

The level- r SA refinement (tightening) $\mathcal{S}^{(r)}(P)$ of P is obtained by projecting $M^{(r)}$ onto $x_\emptyset = 1$, i.e. $\mathcal{S}^{(r)}(P) = M^{(r)} \cap \{x \in \mathbb{R}^{\mathcal{P}_1} : x_\emptyset = 1\}$.

Occasionally we abuse notation and we treat $\mathcal{N}_+^{(r)}(P), \mathcal{S}^{(r)}(P)$ as subsets of $[0, 1]^m$, instead of $\{x \in [0, 1]^{m+1} : x_\emptyset = 1\}$. Also, relaxations derived by LS₊ and SA are in principle incomparable.

We observe that level-1 SA tightening coincides with the level-1 Lovász-Schrijver-LP tightening ($\mathcal{N}_+^{(1)}(P)$ without the PSD constraint). This seemingly weak LP solves the problematic star graph.

► **Proposition 5.** *Let G be the star graph of Observation 1. Then the level-1 SA tightening of $P_t(G)$ has integrality gap 1.*

Proof. Let x be a vector in the level-1 SA tightening of $P_t(G)$, and let y be its moment matrix \mathcal{Y} as in Definition 4. Suppose now that for some $\mathbf{b}, \bar{\mathbf{b}}, \mathbf{d}, \bar{\mathbf{d}} \in \mathbb{R}^n$ and $a \in \mathbb{R}$ we have $\mathcal{Y}\mathbf{e}_\emptyset^T = (1, \mathbf{b}^T, a, \mathbf{d}^T)$ and $\mathcal{Y}\mathbf{e}_{\{n+1\}}^T = (a, \bar{\mathbf{b}}^T, a, \bar{\mathbf{d}}^T)$, where we explicitly assume that the list of indices has first all vertices (with the center being last), followed by all edges. Note that with this terminology, the value of the objective for such a solution is $a + \mathbf{1}_n^T \mathbf{b}$, which we need to compare to $opt = 1$.

Next we focus on $\mathcal{Y}(\mathbf{e}_\emptyset - \mathbf{e}_{\{n+1\}})$ that satisfies all homogenized constraints of $P_t(G)$, and in particular constraints (1) of edges $\{n+1, i\}$, $i = 1, \dots, n$, which require that $\mathbf{b} - \bar{\mathbf{b}} \geq \mathbf{d} - \bar{\mathbf{d}}$. Similarly, constraint (2) of $P_t(G)$ implies that $\mathbf{1}_n^T(\mathbf{d} - \bar{\mathbf{d}}) \geq (1 - a)t$. Therefore $a + \mathbf{1}_n^T \mathbf{b} \geq a + \mathbf{1}_n^T(\mathbf{d} - \bar{\mathbf{d}}) \geq a + (1 - a)t \geq 1 = opt$. ◀

Recall that by Proposition 2 the star graph is also responsible for a n/t integrality gap for the SDP of [15], i.e. the relaxation which the best algorithm known for t -PVC is based on. The surprising conclusion from Proposition 5 is that a simple LP that one can derive systematically from $P_t(G)$ outperforms that particular SDP for a specific instance. This is in contrast to other known examples of level- $\Theta(m)$ LS tightenings that are strictly weaker than natural and static SDP relaxations. Finally, it is worthwhile mentioning that we do not know whether constant-level L&P tightenings of (t -PVC-LP) derive the SDP of [15].

For algorithmic purposes, a number of SA variants have been proposed that give rise to hierarchies of SDPs (see [1] for a list of them). The simplest variation, and the one that has resulted surprisingly strong positive results, is usually referred as the mixed hierarchy. This system, that we denote here by SA_+ imposes an additional PSD constraint.

► **Definition 6 (The SA_+ system).** Let K be a conified polytope $P \subseteq [0, 1]^m$. The level- r SA_+ tightening of K is defined as the refinement of cone $M^{(r)}$, as in Definition 4, where the $(m + 1)$ -leading principal minor of the moment matrix \mathcal{Y} , i.e. the principal minor of \mathcal{Y} that is indexed by sets of variables of size at most 1, is PSD.

Level- r SDPs derived by the SA_+ and LS_+ systems are not comparable. In Section 4 we introduce a further refinement of SA_+ that is strictly tighter than LS_+ , and for which we actually derive the same IG lower bounds as in SA. We postpone its definition due to its technicality.

By the generic algorithmic properties common to LS_+ , SA and SA_+ systems, and for the t -PVC polytope, it is immediate that for any graph $G = (V, E)$ the level- $(|V| + |E|)$ relaxations have integrality gap 1. However, from the proof of convergence from all systems, it easily follows that vectors in level- r relaxations satisfy *any* constraint that is valid for the integral hull of $P_t(G)$ and that has support at most r . If opt denotes the optimal value for $G = (V, E)$ then $\sum_{i \in V} x_i \geq opt$ is a constraint valid for every integral solution with support $|V|$. Hence, level- $|V|$ LPs or SDPs derived by SA, LS_+ and SA_+ systems can solve any t -PVC instance exactly. Can level- r relaxations close the unbounded integrality gap of $P_t(G)$ as exhibited in Observation 1, for $r = o(|V|)$? We answer this question in the negative in the next sections by proving strong integrality gaps for superconstant level LP and SDP relaxations. As a byproduct, we show this way that LPs and SDPs that give rise

to algorithms that run in superpolynomial time cannot solve to any good proximity even the tractable combinatorial problem t -PVC where $t = \Theta(1)$.

3 IG lower bounds for the Sherali-Adams LP system

This section is devoted in proving one of our main results.

► **Theorem 7.** *Let n, r, t be integers with $n \geq 2r + 2t + 2$. Then the integrality gap of the level- r SA-tightening of (t -PVC-LP) on graphs with n vertices is at least $\binom{n-2r}{2}/t \cdot n$.*

For this we fix a clique $G = (V, E)$ on n vertices, along with r, t such that $n \geq 2r + 2t + 2$. We start by presenting Random Process 1, that defines a distribution of 0-1 assignments for variables of the polytope $P_t(G)$.

Random Process 1 (Definition of distribution \mathcal{D}_p)

Require: A fixed $p \in [0, 1]$.

- 1: **for** $i \in V$ **do**
- 2: Independently at random, set $x_i = 1$ with probability p
- 3: **end for**
- 4: **for** $e \in E$ **do**
- 5: Set x_e equal to 1 as long as e is incident to some i for which $x_i = 1$, and otherwise to 0.
- 6: **end for**

Output: Distribution \mathcal{D}_p induced by the experiment above.

We are ready to propose a vector solution $y \in \mathbb{R}^{\mathcal{P}_{r+1}}$ to the level- r SA tightening of $P_t(G)$. For $A \in \mathcal{P}_{r+1}$ (with ground set $V \cup E$), and for each $q \in A$, let X_q be the random variable which equals 1 if $x_q = 1$ in the random experiment of \mathcal{D}_p , and 0 otherwise. For all such $A \subseteq V \cup E$, we define

$$y_A := \mathbb{E}_{\mathcal{D}_p} \left[\prod_{q \in A} X_q \right] = \mathbb{P}_{\mathcal{D}_p} [\forall q \in A, x_q = 1] \quad (4)$$

where the last equality is due to that X_q are 0-1 variables. In particular, this means that for all $i \in V$ and $f \in E$ we have $y_{\{i\}} = p$, $y_{\{f\}} = 2p - p^2$.

We use the following technical lemma; it is a standard observation used in many SA lower bounds.

► **Lemma 8.** *For $Y \cup N \in \mathcal{P}_{r+1}$, let $w_{Y,N} := \sum_{\emptyset \subseteq T \subseteq N} (-1)^{|T|} y_{Y \cup T}$. Then $w_{Y,N} = \mathbb{P}_{\mathcal{D}_p(Y \cup N)} [\forall q \in Y, X_q = 1, \& \forall q' \in N, X_{q'} = 0]$.*

We can now prove that y is solution to the level- r SA polytope of t -PVC, for a proper choice of p .

► **Lemma 9.** *For the complete graph $G = (V, E)$ on n vertices, and for all r, t with $n \geq 2r + 2t + 2$, let $y \in \mathbb{R}^{\mathcal{P}_{r+1}}$ be as in (4), where $p = t/\binom{n-2r}{2}$. Then $y \in S^{(r)}(P_t(G))$.*

Proof. Let $Y, N \in \mathcal{P}_r$ with $|Y \cup N| \leq t$. We need to show that $\bar{y} := \mathcal{Y} \sum_{\emptyset \subseteq T \subseteq N} (-1)^{|T|} \mathbf{e}_{Y \cup T} \in \mathbb{R}^{\mathcal{P}_1}$ satisfies all constraints of $P_t(G)$ (after they are homogenized).

Asking that \bar{y} satisfies the constraint (1) for an edge $e = \{i, j\}$ is the same as asking that $w_{Y \cup \{i\}, N} + w_{Y \cup \{j\}, N} - w_{Y \cup \{e\}, N} \geq 0$. Note that $|Y \cup N \cup \{i, j\}| \leq r + 2$. Due to Lemma (8) and by linearity of expectation we have

$$w_{Y \cup \{i\}, N} + w_{Y \cup \{j\}, N} - w_{Y \cup \{e\}, N} = \mathbb{E}_{\mathcal{D}_p(Y \cup N \cup \{i, j\})} \left[\prod_{q \in Y} X_q \prod_{p \in N} (1 - X_p) (X_i + X_j - X_e) \right].$$

Recall in Random Process 1 we set $x_e = 1$ only when at least one among x_i, x_j is already set to 1. Therefore the previous expected value is always non negative.

In a similar manner we can show that box constraints (3) are satisfied. First, constraints of the form $x_q \geq 0$, $q \in V \cup E$ are satisfied for \bar{y} , since by Lemma 8, $w_{Y \cup \{q\}, N}$ represents a probability of an event. As for constraints $x_q \leq 1$, we need to prove that $w_{Y \cup \{q\}, N} \leq w_{Y, N}$. This is true again due to Lemma 8, and because the event associated with $w_{Y, N}$ is logically implied by that of $w_{Y \cup \{q\}, N}$.

Finally we need to show that \bar{y} satisfies constraint (2), i.e. constraint $\sum_{e \in E} w_{Y \cup \{e\}, N} \geq t \cdot w_{Y, N}$. For this we recall that $|Y \cup N| \leq r$, and so in the original clique on n vertices, there is a subclique $G' = (U, F)$ on at least $n - 2r \geq 4$ vertices, such that no edge in F is incident to any element (vertex or edge) in $Y \cup N$, and $|F| \geq \binom{n-2r}{2} > 0$. This means that for every $f \in F$ the event that $X_f = 1$ is independent to any 0-1 assignment on variables in $Y \cup N$, while $\mathbb{P}_{\mathcal{D}_p}[X_f = 1] = 2p - p^2 \geq p$, since $p = t / \binom{n-2r}{2} \leq t / \binom{2t+2}{2} < 1/2$. Since we also have $|F| \cdot p = |F| \cdot t / \binom{n-2r}{2} \geq t$, we conclude that $\sum_{e \in E} w_{Y \cup \{e\}, N} \geq \sum_{e \in F} w_{Y \cup \{e\}, N} = |F| \cdot p \cdot w_{Y, N} \geq t \cdot w_{Y, N}$, as promised. \blacktriangleleft

The objective of the level- r SA LP is no more than $n \cdot p = t \cdot n / \binom{n-2r}{2}$, while the optimal solution of the input graph has cost 1, concluding the proof of Theorem 7.

It is worthwhile noticing that our superconstant integrality gaps lower bounds hold only for values of parameter $t = o(n)$. The reader can easily verify that when the input is the n -clique, then the optimal solution to (t -PVC-LP) is exactly $t/(n-1)$ (e.g. using the dual of (t -PVC-LP)). Therefore, for any constant c and when $n/c \leq t \leq n-1$, for which the optimal solution to t -PVC is still 1, the integrality gap of (t -PVC-LP) is strictly less than c .

4 IG lower bounds for various SDP hierarchies

4.1 SDPs derived by the SA_+ and LS_+ systems

In this section we argue that the moment matrix \mathcal{Y} of solution y that we proposed in Lemma 9 satisfies very strong PSD conditions. This will immediately imply the same IG lower bounds of Theorem 7 also for stronger SDP systems, as summarized in the next theorem.

► Theorem 10. *Let n, r, t be integers with $n \geq 2r + 2t + 2$. Then the integrality gap of the level- r LS_+ and SA_+ tightenings of (t -PVC-LP) on graphs with n vertices is at least $\binom{n-2r}{2} / t \cdot n$.*

For proving Theorem 10, we fix the clique $G = (V, E)$ on n vertices, together with r, t such that $n \geq 2r + 2t + 2$. In all our arguments below we use $y \in \mathbb{R}^{\mathcal{P}_{r+1}}$ as defined in (4), as well as vector w (indexed by pairs of sets of variables) as it appears in Lemma 8. We also define the matrix $\mathcal{X}^{Y, N} \in \mathbb{R}^{\mathcal{P}_1 \times \mathcal{P}_1}$, which at entry A, B (i.e. any two sets of size at most 1) equals $w_{Y \cup A \cup B, N}$. Note that matrix $\mathcal{X}^{Y, N}$ is exactly the moment matrix of random variables $\{X_q\}_{q \in V \cup E}$ condition on $X_q = 1$ for all $q \in Y$, and $X_{q'} = 0$ for all $q' \in N$, scaled by the constant $\mathbb{P}_{\mathcal{D}_p}[\forall q \in Y, X_q = 1 \ \& \ \forall q' \in N, X_{q'} = 0]$. In particular, for each $q \in V \cup E$ we have that vectors $\mathcal{X}^{Y, N} \mathbf{e}_q, \mathcal{X}^{Y, N} (\mathbf{e}_\emptyset - \mathbf{e}_q)$ satisfy all constraints of $P_t(G)$.

Now recall that $y \in \mathbb{R}^{\mathcal{P}_{r+1}}$ is obtained by the global distribution \mathcal{D}_p that associates any 0-1 assignment of variables of $P_t(G)$ with some probability. In particular, if $x \in \{0, 1\}^{\mathcal{P}_1}$, with $x_\emptyset = 1$, is such a 0-1 assignment, then xx^T is a rank 1 PSD matrix. Clearly, matrix $\mathcal{X}^{Y,N}$ is a convex combination of such rank-1 PSD matrices, hence it is PSD as well. We conclude with an Observation.

► **Observation 11.** *Let Y, N be any subsets of $V \cup E$ such that $|Y \cup N| \leq r - 1$. Then $\mathcal{X}^{Y,N}$ is positive semidefinite.*

It is now immediate that our SA solution y satisfies also the extra PSD constraint imposed by SA_+ . What we only need to observe is that the leading principal minor of \mathcal{Y} indexed by sets of size at most 1 is exactly $\mathcal{X}^{\emptyset, \emptyset}$, which is PSD by Observation 11. Hence, Theorem 7 also holds when SA tightenings are replaced by SA_+ tightenings.

Next we argue that our SA solution is robust against much stronger SDP refinements. Note that vector w is well defined for all level- r SA solutions y . Especially when y is obtained as a convex combination of integral vectors, all matrices $\mathcal{X}^{Y,N}$ are PSD, for all $|Y \cup N| \leq r - 1$. That is, the latter constraints constitute a further refinement of the SA_+ system. Again by Observation 11 it is immediate that our level- r SA solution fools also these exponentially many (in r) PSD conditions. These new PSD refinements are stronger than the constraints derived by the level- $(r - 1)$ LS_+ system (see [30]); this concludes the proof of Theorem 10.

4.2 On SDPs derived by the Lasserre system

A natural question to ask is whether our SA solution fools SDPs derived by the so-called Lasserre (La) system [21]. The level- r La-SDP is defined as follows. For $y \in \mathbb{R}^{2r+2}$, the La-moment matrix \mathcal{Z} is a matrix indexed by \mathcal{P}_{r+1} with $\mathcal{Z}_{A,B} = y_{A \cup B}$. For each constraint $\sum_i \alpha_i^{(l)} x_i - \beta^{(l)} \geq 0$ of P , the La-slack moment matrix $\mathcal{Z}^{(l)}$ is a matrix indexed by \mathcal{P}_r with $\mathcal{Z}_{A,B}^{(l)} = \sum_i \alpha_i^{(l)} y_{A \cup B \cup \{i\}} - \beta^{(l)} y_{A \cup B}$. The level- r La SDP requires that all matrices \mathcal{Z} and $\{\mathcal{Z}^{(l)}\}_l$ are PSD. Notably, the PSDness of proper principal minors of matrices \mathcal{Z} and $\{\mathcal{Z}^{(l)}\}_l$ imply the level- r SA linear constraints [22]. As such, the level- r La SDP is at least as strong as the level- r SA LP. Unfortunately, we show that the level-1 La SDP is not fooled by our SA solution.

► **Lemma 12.** *For any constant r , the level-1 La SDP eliminates the level- r solution proposed in Lemma 9.*

Proof. Fix n, t, p , and let y be the solution to the level- (r) SA-tightening as described in Lemma 9. Consider the level-1 slack matrix for (2). In order to prove that this matrix is not PSD, it suffices to focus on its principal minor $\bar{\mathcal{Z}}$ that is indexed only by subsets of vertices. To that end, let $y_A \in \mathbb{R}^{\mathcal{P}_1}$ be the indicator vector of set $A \subseteq V$. Let also S_n denote the expected slack we have in constraint (2) when each vertex is chosen with probability p in the n -clique, and $C_{n,a}$ be the number of edges that are covered by choosing a many vertices in the same graph. Then, it is easy to verify by definition that $\bar{\mathcal{Z}}$ has the form $(\bar{\mathcal{Z}})_{I,J} = p^{|I \cup J|} (S_{n-|I \cup J|} + C_{n,|I \cup J|})$. Applying the Schur complement on $\bar{\mathcal{Z}}$ with respect to the entry $(\bar{\mathcal{Z}})_{\emptyset, \emptyset} = S_n$, and given that $S_n > 0$, we have that $\bar{\mathcal{Z}}$ is PSD if and only if $M - \frac{(p(S_{n-1} + C_{n,1}))^2}{S_n} J_n$ is PSD, where M is the minor of $\bar{\mathcal{Z}}$ indexed by sets of vertices of size 1, and J_n is the all-one $n \times n$ matrix. By symmetry, all rows of M have the same sum, i.e. the all-one vector $\mathbf{1}$ is an eigenvector for the Schur complement. Elementary calculations then show that the leading term of the corresponding eigenvalue, when $p = c/n^2$, is $(-2c^4 - \frac{15c^3}{2} - 2c^2) \frac{1}{n} < 0$ (the rest of the summands are of order $o(1/n)$). ◀

5 Discussion / Open Problems

The algorithmic significance of our results pose a natural (and classic) open problem, related also to questions on extended formulations; Does t -PVC admit a polysize (or tractable) LP or SDP relaxation that has integrality gap no more than 2, even when $t = O(n)$? It is notable that this question has been studied in [3] for a generalization of t -PVC but with no implications to our problem. Note also that our strongest IG lower bounds are valid only when $t/n = \epsilon$, for small enough $\epsilon > 0$, where n is the number of vertices of the input graph. As a result, another interesting open question is, given t and n , find the smallest $r = r(n, t)$ for which the level- r LP or SDP derived by some L&P system has integrality gap no more than 2. In particular, can it be that $r = \omega(1)$ when $t \geq n$?

Finally, our SDP IG lower bounds make explicit that global distributions of 0-1 assignments can be used to witness solutions to SA LP tightenings of superconstant integrality gaps. We also demonstrate that it is almost straightforward to show that the same solutions are robust against SDP tightenings of many L&P systems except the La system. Can the same family of global distributions fool La SDPs when it is also enriched with intuitive and stronger conditions? A generic positive or negative answer would give new insights in understanding the power of the various SDP hierarchies.

Acknowledgments. We would like to thank the anonymous referees for their valuable comments.

References

- 1 Y. Au and L. Tunçel. A comprehensive analysis of polyhedral lift-and-project methods. *CoRR*, abs/1312.5972, 2013.
- 2 S. Benabbas, S. O. Chan, K. Georgiou, and A. Magen. Tight Gaps for Vertex Cover in the Sherali-Adams SDP Hierarchy. In *FSTTCS*, volume 13 of *LIPICs*, pages 41–54, 2011.
- 3 S. K. Bera, S. Gupta, A. Kumar, and S. Roy. Approximation algorithms for the partition vertex cover problem. In *WALCOM*, volume 7748 of *LNCS*, pages 137–145. Springer, 2013.
- 4 N. H. Bshouty and L. Burroughs. Messaging a linear programming solution to give a 2-approximation for a generalization of the vertex cover problem. In *STACS*, volume 1373 of *LNCS*, pages 298–308. Springer, 1998.
- 5 B. Caskurlu and K. Subramani. On partial vertex cover on bipartite graphs and trees. *CoRR*, abs/1304.5934, 2013.
- 6 M. Charikar, K. Makarychev, and Y. Makarychev. Integrality gaps for Sherali-Adams relaxations. In *STOC*, pages 283–292, New York, NY, USA, 2009. ACM Press.
- 7 K. K. H. Cheung. Computation of the lasserre ranks of some polytopes. *Math. Oper. Res.*, 32(1), 2007.
- 8 E. Chlamtáč and M. Tulsiani. Convex relaxations and integrality gaps. In Miguel F. Anjos and Jean B. Lasserre, editors, *Handbook on Semidefinite, Conic and Polynomial Optimization*, volume 166 of *International Series in Operations Research & Management Science*, pages 139–169. Springer US, 2012.
- 9 W. Cook and S. Dash. On the matrix-cut rank of polyhedra. *Mathematics of Operations Research*, 26(1):19–30, 2001.
- 10 Irit Dinur and Shmuel Safra. On the hardness of approximating minimum vertex-cover. *Annals of Mathematics*, 162(1):439–486, 2005.
- 11 R. Gandhi, S. Khuller, and A. Srinivasan. Approximation algorithms for partial covering problems. *J. Algorithms*, 53(1):55–84, 2004.

- 12 K. Georgiou and E. Lee. Lift and project systems performing on the partial-vertex-cover polytope. *CoRR*, abs/1409.6365v1, 2014.
- 13 K. Georgiou and A. Magen. Expansion Fools the Sherali-Adams System: Compromising Local and Global Arguments. Technical Report CSRG-587, University of Toronto, November 2008.
- 14 K. Georgiou, A. Magen, T. Pitassi, and I. Tourlakis. Integrality gaps of $2-o(1)$ for vertex cover SDPs in the Lovász-Schrijver hierarchy. *SIAM J. Comput.*, 39(8):3553–3570, 2010.
- 15 E. Halperin and A. Srinivasan. Improved approximation algorithms for the partial vertex cover problem. In *APPROX*, volume 2462, pages 161–174. Springer, 2002.
- 16 D.S. Hochbaum. The t-vertex cover problem: Extending the half integrality framework with budget constraints. In *APPROX*, volume 1444 of *LNCS*, pages 111–122. Springer Berlin Heidelberg, 1998.
- 17 C. Kenyon-Mathieu and W. F. de la Vega. Linear programming relaxations of maxcut. In *SODA*, pages 53–61. ACM Press, 2007.
- 18 S. Khot and O. Regev. Vertex cover might be hard to approximate to within $2-\epsilon$. *J. Comput. Syst. Sci.*, 74(3):335–349, 2008.
- 19 S.G. Kolliopoulos and Y. Moysoglou. Sherali-Adams gaps, flow-cover inequalities and generalized configurations for capacity-constrained Facility Location. In *APPROX*, LNCS, page to appear, 2014.
- 20 J. Könemann, O. Parekh, and D. Segev. A unified approach to approximating partial covering problems. *Algorithmica*, 59:489–509, 2011.
- 21 J.B. Lasserre. An explicit exact SDP relaxation for nonlinear 0-1 programs. In *IPCO*, volume 2081 of *LNCS*, pages 293–303. Springer, Berlin, 2001.
- 22 M. Laurent. A comparison of the Sherali-Adams, Lovász-Schrijver, and Lasserre relaxations for 0-1 programming. *Math. Oper. Res.*, 28(3):470–496, 2003.
- 23 L. Lovász and A. Schrijver. Cones of matrices and set-functions and 0-1 optimization. *SIAM Journal on Optimization*, 1(2):166–190, May 1991.
- 24 M. Mastrolilli. The lasserre hierarchy in almost diagonal form. *CoRR*, abs/1312.6493, 2013.
- 25 J. Mestre. A primal-dual approximation algorithm for partial vertex cover: Making educated guesses. *Algorithmica*, 55(1):227–239, 2009.
- 26 G. Schoenebeck. Linear level lasserre lower bounds for certain k-CSPs. In *FOCS*, pages 593–602. IEEE Computer Society, 2008.
- 27 G. Schoenebeck, L. Trevisan, and M. Tulsiani. Tight integrality gaps for Lovász-Schrijver LP relaxations of vertex cover and max cut. In *STOC*, pages 302–310. ACM Press, 2007.
- 28 H.D. Sherali and W.P. Adams. A hierarchy of relaxations between the continuous and convex hull representations for 0–1 programming problems. *SIAM J. Discrete Math.*, 3(3):411–430, 1990.
- 29 A. Srinivasan. Distributions on level-sets with applications to approximation algorithms. In *FOCS*, pages 588–599, 2001.
- 30 Iannis Tourlakis. *New lower bounds for approximation algorithms in the Lovász-Schrijver hierarchy*. PhD thesis, Department of Computer Science, June 2006.
- 31 J. Tu, J. Du, and F. Yang. An iterative rounding 2-approximation algorithm for the k-partial vertex cover problem. *Acta Mathematicae Applicatae Sinica, English Series*, 30(2):271–278, 2014.
- 32 M. Tulsiani. CSP gaps and reductions in the Lasserre hierarchy. In *STOC*, pages 303–312, New York, NY, USA, 2009. ACM Press.