Semilattices of Fault Semiautomata

J. A. Brzozowski[1] and H. Jürgensen[2,3]

[1] Department of Computer Science, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1; email: brzozo@uwaterloo.ca.
[2] Department of Computer Science, The University of Western Ontario, London, Ontario, Canada, N6A 5B7; email: helmut@uwo.ca.
[3] Institut für Informatik, Universität Potsdam, Am Neuen Palais 10, D–14469 Potsdam, Germany; email: helmut@cs.uni-potsdam.de.

Abstract. We study defects affecting state transitions in sequential circuits. The fault-free circuit is modeled by a semiautomaton M, and 'simple' defects, called *single faults*, by a set $S = \{M^1, ..., M^k\}$ of 'faulty' semiautomata. To define *multiple* faults from S, we need a binary composition operation, say $\odot$, on semiautomata, which is idempotent, commutative, and associative. Thus, one has the free semilattice $S^\odot$ generated by S. In general, however, the single faults are not independent; a finite set E of equations of the form $M^{i_1} \odot \ldots \odot M^{i_h} = M^{j_1} \odot \ldots \odot M^{j_k}$ describes the relations among them. The pair (S, E) is a finite presentation of the quotient semilattice $S^\odot/\eta$, where η is the smallest semilattice congruence containing E. In this paper, we first characterize such abstract quotient semilattices. We then survey the known results about random-access memories (RAMs) for the Thatte-Abraham fault model consisting of stuck-at, transition, and coupling faults. We present these results in a simplified semiautomaton model and give new characterizations of two fault semilattices.

1 Single and Multiple Faults

In this paper we study the following general problem. We have a *fault-free object*, which is completely abstract at this stage. When the object is manufactured, zero or more 'simple defects' may be introduced. Each of these defects produces a faulty version of the object, called a *single fault*. Let $D = \{d_1, \ldots, d_k\}$ be the set of simple defects and let $S = \{s_1, \ldots, s_k\}$, $k \geq 0$, be the corresponding set of single faults.

If several simple defects are present in the faulty object at the same time, we have a *multiple fault*. Our objective is to characterize such multiple faults in terms of the single faults.

In the simplest case, the single faults are not related in any way. Here, a multiple fault is simply a set of single faults. For example, suppose there are two single faults. The fault-free object corresponds to the empty set of single faults; the objects containing one defect, to $\{s_1\}$ and $\{s_2\}$; and the double fault, to the set $\{s_1, s_2\}$. The composition operation on single faults that yields multiple faults is set union. Our multiple fault model is the algebra $(2^S, \cup, \emptyset)$. Note that this algebra is a finite semilattice, with identity $\emptyset$ and zero S, generated by the finite set S under the union operation.

In practice a set A of defects may result in a faulty object that is indistinguishable from the faulty object corresponding to defects from a different set B. We will write $A \equiv B$ to indicate that A and B result in the same object. Thus, in general, we have a finite set S of single faults, the set 2^S of potential multiple faults, and a finite set $\mathcal{E} = \{(A \equiv B) \mid A, B \in 2^S\}$ of equations. Our model for multiple faults is the quotient semilattice $2^S/\eta_{\mathcal{E}}$, where $\eta_{\mathcal{E}}$ is the smallest semilattice congruence containing $\mathcal{E}$.

2 Finitely Presented Semilattices

Abstractly we have the following setting for our problem. We are given a finite set S of generators and a finite set E of equations of the form $a \equiv b$ where a and b are words over the alphabet S. Let S^* be the free monoid with identity 1 generated by S, and let η be the smallest semilattice congruence on S^* containing E. Then the quotient semilattice $S^\bullet = S^*/\eta$ is the semilattice defined by the presentation (S, E). For any word u over S, let $[u]_\eta$ be the η-class of u. The multiplication $\bullet$ on $S^\bullet$ is given by $[u]_\eta \bullet [v]_\eta = [uv]_\eta$.

For a word u over S, let $\alpha(u)$ be the set of elements of S appearing in u. If u and v are words over S such that $\alpha(u) = \alpha(v)$, then u and v represent the same element of the free semilattice generated by S, hence also of $S^\bullet$. Thus, via α, the free semilattice with identity generated by S is isomorphic with the semilattice $(2^S, \cup, \emptyset)$. For an equation $a \equiv b$ in E, let $\alpha(a \equiv b)$ be the equation $\alpha(a) \equiv \alpha(b)$; moreover, let $\alpha(E)$ be the set of equations $\alpha(a \equiv b)$ where $a \equiv b$ is in E. Let θ be the smallest congruence relation on 2^S containing $\alpha(E)$. Then the semilattice $S^\bullet$ is isomorphic with the quotient semilattice $2^S/\theta$. Thus to compute the product of two elements in $S^\bullet$, it is equivalent to compute the product of the corresponding sets in $2^S/\theta$.

In general, there may be several different subsets of S representing the same element of $2^S/\theta$. However, for every $T \subseteq S$ one can choose a canonical representative as follows.

Proposition 1. *Let $T \subseteq S$. Then there is a unique set rep(T) in the θ-class $[T]_\theta$ of T such that $T' \subseteq$ rep(T) for all $T' \in [T]_\theta$.*

Proof. The class $[T]_\theta$ is closed under union. Hence, the union of all sets $T' \in [T]_\theta$ is the desired canonical representative.

We now give an algorithm for constructing the canonical representative *rep*(T) of any set $T \subseteq S$. Given that α is a semilattice isomorphism, instead of words and equations on words, we consider only subsets of S and equations on sets. An equation $e_i = A_i \equiv B_i$ is *applicable* to T if $A_i \subseteq T$ or $B_i \subseteq T$. The set of equations will be represented by an array $\mathcal{E} = (e_1, \ldots, e_m)$. Variable $\mathcal{E}'$ represents equations that have not yet been eliminated. Variable T' represents the set obtained from T by the equations used so far. The size of the array $\mathcal{E}'$ is denoted by $|\mathcal{E}'|$. An *application* of an applicable equation e_i to T' consists of replacing T' by $T' \cup A_i \cup B_i$ and of deleting e_i from $\mathcal{E}'$. With these assumptions we have the algorithm shown in Fig. 1.

Fig. 1. Function *rep*. Scoping is indicated by indentation.

function $rep(\mathcal{E}$:**array**$[1..m]$ **of** equations, T: subset of S): subset of S;
var $\mathcal{E}'$: **array**$[1..m]$ **of** equations, T': subset of S;
$\mathcal{E}' \leftarrow \mathcal{E}$; $T' \leftarrow T$;
while $\mathcal{E}' \neq \emptyset$ **do**
    $i \leftarrow 1$;
    while e_i not applicable **do**
        $i \leftarrow i + 1$;
        if $i > |\mathcal{E}'|$ **then**
            $rep \leftarrow T'$;
            exit $\{rep\}$;
    $\mathcal{E}' \leftarrow \mathcal{E}'$ with e_i deleted; $T' \leftarrow T' \cup A_i \cup B_i$;
$rep \leftarrow T'$;

Let $S^\vee$ be the set of distinct representatives obtained by *rep*. Let $\vee$ be defined as follows: For any sets $A, B \in S^\vee$, $A \vee B = rep(A \cup B)$.

Theorem 1. *The algebra* $(S^\vee, \vee, \emptyset)$ *is a semilattice with identity. The three semilattices* $S^\bullet$, $2^S/\theta$ *and* $S^\vee$ *are isomorphic.*

Proof. The mapping *rep* from 2^S to $S^\vee$ induces an isomorphism μ from $2^S/\theta$ onto $2^\vee$ by $\mu([T]_\theta) = rep(T)$ for $T \subseteq S$.

Note that Theorem 1 provides an algorithm for solving the word problem in finitely presented semilattices. The solvability of the word problem itself follows already from some very general theorems about finitely presented algebras [4]. Surprisingly, however, we failed to find our simple solution to this problem in the literature.

As in any finite semilattice, one can derive a second semilattice operation $\wedge$ on $S^\vee$ such that $(S^\vee, \vee, \wedge, \emptyset, S)$ is a lattice, where

$$T \wedge T' = \bigvee_{U \in S^\vee, U \subseteq T \cap T'} U.$$

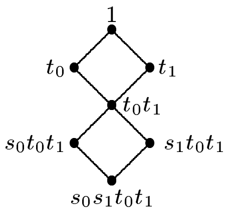
It turns out that $T \wedge T' = T \cap T'$, that is, the set of canonical representatives is closed under intersection.

Example 1. Let $S = \{s_0, s_1, t_0, t_1\}$, and

$$E = \{s_i t_j \equiv s_i \mid i, j \in \{0, 1\}\} \cup \{s_0 s_1 \equiv s_0 s_1 t_0 t_1\}.$$

One verifies that $S^\bullet = S^*/\eta$ has the 7 elements $\emptyset$, t_0, t_1, $t_0 t_1$, $s_0 t_0 t_1$, $s_1 t_0 t_1$, and $s_0 s_1 t_0 t_1$. The Hasse diagram of $S^\bullet$ is shown in Fig. 2.

Fig. 2. Semilattice of Example 1



3 Faults in Sequential Circuits

We now make our problem more concrete by considering sequential circuits. Such circuits can be modeled by Mealy automata defined as follows. A *finite deterministic Mealy automaton* is a quintuple $M = (Q, X, Y, \delta, \lambda)$, where Q is a finite nonempty set called the *state set*, X is a finite nonempty set called the *input alphabet*, δ is a function, called the *transition function*, from $Q \times X$ into Q, set Y is a finite nonempty set called the *output alphabet*, and λ is a function, called the *output function*, from $Q \times X$ into Y. When a defect is present in a sequential circuit, we assume that the faulty circuit is still represented by an automaton. A faulty automaton may differ from the fault-free one in the state set, transition function, and output function.

In this paper we consider only defects affecting the state set and the transition function; hence, we can use semiautomata. A *finite deterministic semiautomaton* is denoted by $M = (Q, X, \delta)$, where Q, X, and δ are as above. The *empty semiautomaton* is $\mathbf{0} = (\emptyset, X, \emptyset)$.

Example 2. We give an example involving partial semiautomata, where the transition function is defined only on a subset of $Q \times X$. Consider an arbitrary partial semiautomaton $M = (Q, X, \delta)$ and the set S of faulty partial semiautomata of the following kind. For each transition $\delta(q, x)$ of M there is a faulty semiautomaton $M_{q,x} = (Q, X, \delta_{q,x})$, such that $\delta_{q,x}$ coincides with δ, except that $\delta_{q,x}(q, x)$ is undefined. The composition of two partial semiautomata $M^1 = (Q, X, \delta^1)$ and $M^2 = (Q, X, \delta^2)$ with $\delta^1, \delta^2 \subseteq \delta$, is $M^{1\odot 2} = M^1 \odot M^2 = (Q, X, \delta^{1\odot 2})$, where $\delta^{1\odot 2} = \delta^1 \cap \delta^2$. Clearly, M is the identity element, and the semiautomaton $(Q, X, \emptyset)$ with no transitions is the zero element. The algebra $(S^{\odot}, \odot, M)$ is isomorphic with the intersection semilattice of the subsets of δ. Note that this is the free semilattice with identity generated by S.

There has been extensive research on products of automata with output [5]. However, very little is known about composition operations on *semiautomata* in general. Hence, in the rest of the paper we turn to semiautomata with more structure, where several results about multiple faults are known.

4 Single Faults in RAMs

A *component semiautomaton* is a finite deterministic semiautomaton $M = (Q, X, \delta)$, where Q has the following property: There exists an integer $n > 0$ and n finite nonempty sets $Q_1, \ldots, Q_n$ such that $Q \subseteq Q_1 \times \ldots \times Q_n$. Thus each state $q \in Q$ is an n-tuple $q = (q_1, \ldots, q_n)$, where $q_i \in Q_i$ for $i = 1, \ldots, n$. In the sequel, Q_i is assumed to be minimal for each i, that is, for every $a \in Q_i$, there is $q \in Q$ such that $q_i = a$.

A component semiautomaton is *binary* if $Q_1, \ldots, Q_n$ are subsets of $\{0, 1\}$. In this paper we consider only binary semiautomata.

We use semiautomata to model random access memories (RAMs). We assume that a RAM consists of n one-bit cells. The *fault-free* n-cell RAM is a semiautomaton $M = (Q, X, \delta)$, where $Q = \{0, 1\}^n$, $X = \bigcup_{i=1}^{n} X_i$ with $X_i = \{w_0^i, w_1^i\}$, and δ is defined by $\delta((q_1, \ldots, q_i, \ldots, q_n), w_a^i) = (q_1, \ldots, a, \ldots, q_n)$, for $a \in \{0, 1\}$. Here $q = (q_1, \ldots, q_n)$ is the present state of the RAM, and $\delta(q, x)$ is the next state of the RAM after input $x \in X$ has been applied. Inputs w_0^i and w_1^i are the 'write 0 to cell i' and the 'write 1 to cell i' operations, respectively[1].

In the sequel, the number n of cells in a RAM is a fixed integer. Also, since all the semiautomata considered here are over the alphabet X given above, this alphabet will not be explicitly mentioned, and a semiautomaton will be denoted simply by $M = (Q, \delta)$. One-cell and two-cell RAMs are shown in Figs. 3 and 4, respectively; they are labelled M.

When a RAM is manufactured, a defect resulting in a faulty RAM may occur. Many defects that occur in practice can still be represented by normal semiautomata [3]. If that is not the case, however, we represent the defective RAM by the special empty semiautomaton $\mathbf{0}$.

To model defective RAMs, we associate with the fault-free n-cell RAM $M = (Q, \delta)$, a family Δ of semiautomata $M' = (Q', \delta')$, called *delta faults* or simply *faults* of M, which may differ from M in the state set and the transition function. The state set Q' is a subset of Q and the transition function δ' is arbitrary. Both M and $\mathbf{0}$ satisfy the definition of a fault.

We now give some examples of common faults: the transition, stuck-at, and coupling faults. These faults constitute the set $\mathcal{T}$ of *Thatte-Abraham faults* [7], and are the main object of our investigation here.

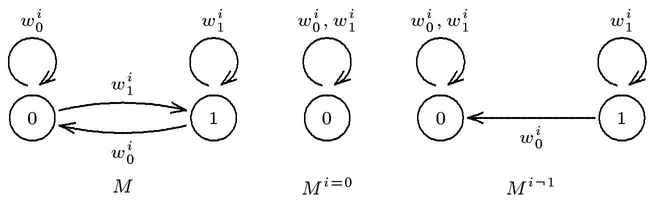
The complement of $a \in \{0, 1\}$ is denoted by $\overline{a}$. A RAM with a *transition* fault in cell i is a fault $M^{i\neg a} = (Q, \delta^{i\neg a})$, where $a \in \{0, 1\}$ and $\delta^{i\neg a}$ behaves like δ *except* that, when the input w_a^i is applied to any state in which cell i is in state $\overline{a}$, the write operation is unsuccessful, that is,

$$\delta^{i\neg a}((q_1, \ldots, q_{i-1}, \overline{a}, q_{i+1}, \ldots, q_n), w_a^i) = (q_1, \ldots, q_{i-1}, \overline{a}, q_{i+1}, \ldots, q_n).$$

Figure 3 shows the transition fault $M^{i\neg 1}$. All operations to other cells are correct and not shown. An n-cell RAM has $2n$ transition faults.

[1] A RAM also has read operations, an output alphabet, and an output function, but these concepts are irrelevant to this paper.

Fig. 3. Transition and stuck-at faults of a one-cell RAM M.

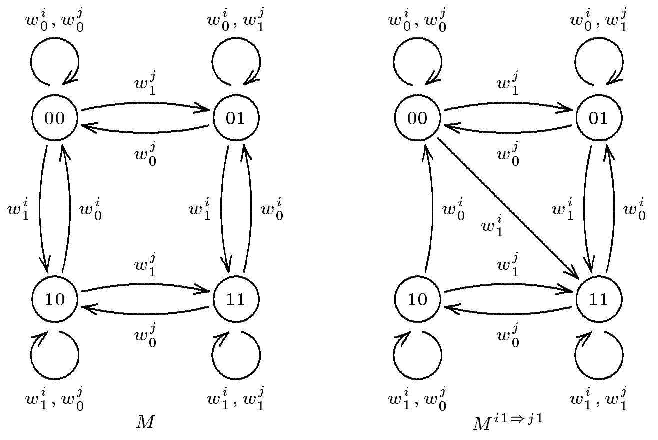


For $a \in \{0,1\}$, a *stuck-at-a* fault in cell i is a fault $M^{i=a} = (Q^{i=a}, \delta^{i=a})$, where $Q^{i=a} = \{q \mid q = (q_1, \ldots, q_{i-1}, a, q_{i+1}, \ldots, q_n)\}$, and $\delta^{i=a}$ is the restriction of δ to $Q^{i=a} \times X$, *except* for the transitions under input $w^i_{\overline{a}}$. This input fails to write $\overline{a}$ in cell i, that is,

$$\delta^{i=a}((q_1, \ldots, q_{i-1}, a, q_{i+1}, \ldots, q_n), w^i_{\overline{a}}) = (q_1, \ldots, q_{i-1}, a, q_{i+1}, \ldots, q_n).$$

Figure 3 illustrates the stuck-at fault $M^{i=0}$. All operations to other cells are correct and not shown. An n-cell RAM has $2n$ stuck-at faults.

Fig. 4. Coupling fault $M^{i1\Rightarrow j1}$ in a two-cell RAM M. The first state component refers to cell i, the second one to cell j.



A *coupling* fault involves two cells, the *coupling* cell i and the *coupled* cell j. When the present states of cells i and j are $\overline{a}$ and $\overline{b}$, respectively, where $a, b \in \{0,1\}$, and a is written into cell i, then cell j, as well as cell i, changes state. The fault is denoted $M^{ia\Rightarrow jb}$. Figure 4 illustrates the fault $M^{i1\Rightarrow j1}$; only the states of cells i and j are shown. All operations to other cells are correct and not shown. An n-cell RAM has $4n(n-1)$ coupling faults.

5 Multiple Faults in RAMs

A 'simple' defect represented by a semiautomaton is considered to be a *single* fault. For example, each transition, stuck-at, and coupling fault is considered to be a single fault in the literature on circuit testing. Suppose now that two defects D^1 and D^2 are present in a RAM, and are represented by semiautomata M^1 and M^2. It should be possible to represent the simultaneous presence of both defects by another semiautomaton. What is needed is an *composition operation on semiautomata,* say $\odot$, such that $M^1 \odot M^2$ would correctly represent the double fault. Although there are cases where it is quite obvious how to construct the semiautomaton for the double fault composed of M^1 and M^2, in general this problem is very difficult.

We make a number of assumptions about the nature of faults. First, whenever the presence of two defects, represented by semiautomata M^1 and M^2, leads to a circuit outside the normal semiautomaton model, we represent the resulting circuit by $\mathbf{0}$; thus $M^1 \odot M^2 = \mathbf{0}$. Second, we assume that the composition of any fault M' with a fault outside the semiautomaton model is outside the model, that is, $\mathbf{0} \odot M' = M' \odot \mathbf{0} = \mathbf{0}$. Third, if defects D^1 and D^2 are represented by the same semiautomaton M', the composite semiautomaton should be exactly the same as each of the two components. Thus, the composition should be *idempotent*. Fourth, the order in which we consider two defects when they are both present in the same circuit should be immaterial. Thus, the composition should be *commutative*. Fifth, the order in which we compose three defects should not matter; the final result should be a circuit with all three defects. Thus, the composition should be *associative*. Fault composition was first considered in [1]. Major results related to this paper were presented in [2,3]. For further justification of the assumptions above see [3].

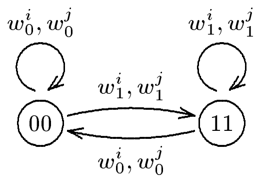
Altogether, for any given set of single faults, we have the *generic* hypothesis that the composition operation should be a semilattice operation, and that the semilattice of multiple faults may contain a zero element $\mathbf{0}$, and, by convention, contains the identity element M.

We now attack the composition problem for the Thatte-Abraham faults. A set $Q' \subseteq Q_1 \times \ldots \times Q_n$ is called a *subproduct* if $Q' = Q'_1 \times \cdots \times Q'_n$, for some sets $Q'_1 \subseteq Q_1, \ldots, Q'_n \subseteq Q_n$. A semiautomaton is called a *subproduct* semiautomaton if its state set is a subproduct. Note that all the faults in $\mathcal{T}$, including M and $\mathbf{0}$ are (binary) subproduct semiautomata. Let $\mathcal{P}$ be the set of all subproduct faults; from now on, we consider only faults from $\mathcal{P}$. An example of a fault that is not in $\mathcal{P}$ is the stuck-equal fault of Fig. 5.

5.1 The Semilattice $\mathcal{T}^\circ$

In addition to the generic properties of fault composition, there are also conditions derived from the physical nature of defects. We now state a set of such *physical* conditions for the Thatte-Abraham fault model. Condition ρ_1

Fig. 5. Stuck-equal fault $M^{i=j}$.



states that a cell that is both stuck-at-0 and stuck-at-1 results in a fault that is outside the semiautomaton model. Condition ρ_2 affirms that a stuck-at fault in a cell is stronger than any transition fault in that cell. Condition ρ_3 declares that a stuck-at fault in a cell is stronger than any coupling fault in which that cell is the coupled cell. For a justification of these conditions see [3,6].

(ρ_1) $M^{i=a} \odot M^{i=\overline{a}} = \mathbf{0}$,
(ρ_2) $M^{i=a} \odot M^{i\neg b} = M^{i=a}$,
(ρ_3) $M^{i=a} \odot M^{jb\Rightarrow ic} = M^{i=a}$.

Our main problem now is to find a binary composition operation on semiautomata that satisfies both the generic conditions and the physical conditions above. Let $M = (Q, \delta)$ be the fault-free n-cell RAM, and let $M^1 = (Q^1, \delta^1)$ and $M^2 = (Q^2, \delta^2)$ be two faults of M. We define the composition $M^1 \circ M^2$ to be the semiautomaton $M^{1\circ 2} = (Q^{1\circ 2}, \delta^{1\circ 2})$, where $Q^{1\circ 2} = Q^1 \cap Q^2$, and $\delta^{1\circ 2}$ is defined as follows. Consider a state $q = (q_1, \ldots, q_n) \in Q^{1\circ 2}$ and an input $x \in X_i$. Let $\delta(q, x) = (p_1, \ldots, p_n)$, $\delta^1(q, x) = (p_1^1, \ldots, p_n^1)$, $\delta^2(q, x) = (p_1^2, \ldots, p_n^2)$, and $\delta^{1\circ 2}(q, x) = (p_1^{1\circ 2}, \ldots, p_n^{1\circ 2})$. To determine $\delta^{1\circ 2}$ one applies the rule:

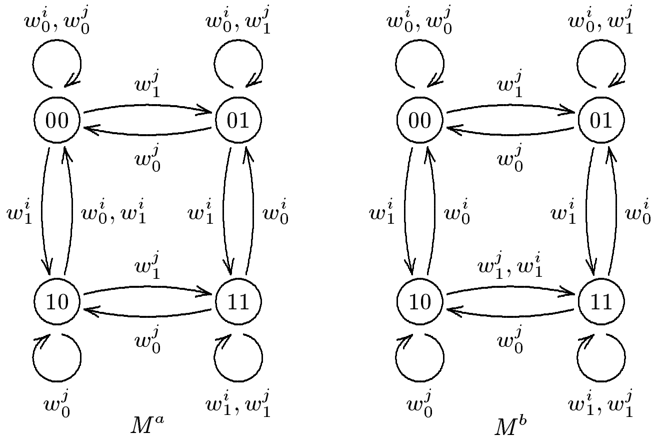
Circle Rule. For all j, (a) $p_j^{1\circ 2} = a$, if $Q_j^{1\circ 2} = \{a\}$, and (b) $p_j^{1\circ 2} = p_j \oplus [(p_j \oplus p_j^1) + (p_j \oplus p_j^2)]$, if $Q_j^{1\circ 2} = \{0, 1\}$, where $+$ and $\oplus$ denote the inclusive and exclusive OR operations, respectively.

In the following theorem, $(\mathcal{T}^\circ, \circ)$ denotes the semilattice generated by $\mathcal{T}$ under the operation $\circ$.

Theorem 2. *The algebra $(\mathcal{P}, \circ)$ is a semilattice with* $\mathbf{0}$ *as zero element and M as the identity, and $\circ$ satisfies ρ_1–ρ_3. The algebra $(\mathcal{T}^\circ, \circ)$ is a subsemilattice of $(\mathcal{P}, \circ)$, and ρ_1–ρ_3 form a minimal set of defining relations for $\mathcal{T}^\circ$. Furthermore, $\mathcal{T}^\circ$ is a proper subsemilattice of $\mathcal{P}$.*

Proof. The first two claims of the theorem were proved in [2]. In that paper, delta faults were defined more narrowly, since they were required to have state sets which were subproducts of Q. Thus the family of delta faults in [2] is precisely $\mathcal{P}$. Semiautomaton M^a of Fig. 6 is not in $\mathcal{T}^\circ$. This follows from the characterization of $\mathcal{T}^\circ$ given in Theorem 4 in Section 6.

Fig. 6. Faults M^a and M^b.



5.2 The Semilattice $\mathcal{T}^\diamond$

A different circuit implementation of a RAM may lead to a different set of physical conditions. In fact, in another useful fault model the composition must satisfy, in addition to ρ_1–ρ_3, also ρ_4 and ρ_5, where

(ρ_4) $M^{i\neg a} \odot M^{ia \Rightarrow jb} = M^{i\neg a}$,
(ρ_5) $M^{i=a} \odot M^{ib \Rightarrow jc} = M^{i=a}$.

The fourth condition requires that a transition fault $M^{i\neg a}$ be stronger than any coupling fault $M^{ia \Rightarrow jb}$. The fifth condition states that the stuck-at fault $M^{i=a}$ is stronger than any coupling fault $M^{ib \Rightarrow jc}$. We again refer the reader to [3,6] for the justification of these conditions. One can verify that ρ_5 is a consequence of ρ_1–ρ_4.

We now define a second composition operation $\diamond$ as follows. Let $M = (Q, \delta)$ be the fault-free n-cell RAM, and let $M^1 = (Q^1, \delta^1)$ and $M^2 = (Q^2, \delta^2)$ be two faults of M. We define the composition $M^1 \diamond M^2$ to be the semiautomaton $M^{1\diamond 2} = (Q^{1\diamond 2}, \delta^{1\diamond 2})$, where $Q^{1\diamond 2} = Q^1 \cap Q^2$ and $\delta^{1\diamond 2}$ is defined as follows. Consider a state $q = (q_1, \ldots, q_n) \in Q^{1\diamond 2}$ and an input $x \in X_i$. Use the same notation as in the Circle Rule, and let $\delta^{1\diamond 2}(q, x) = (p_1^{1\diamond 2}, \ldots, p_n^{1\diamond 2})$. To determine $\delta^{1\diamond 2}$ one applies the rule:

Diamond Rule. If $p_i^1 = p_i^2 = p_i$, then (a) for all j including i, $p_j^{1\diamond 2} = a$, if $Q_j^{1\diamond 2} = \{a\}$, and (b) $p_j^{1\diamond 2} = p_j \oplus [(p_j \oplus p_j^1) + (p_j \oplus p_j^2)]$, if $Q_j^{1\diamond 2} = \{0, 1\}$. Otherwise, $\delta^{1\diamond 2}(q, x) = q$.

The algebra $(\mathcal{P}, \diamond)$ is an idempotent and commutative grupoid with $\mathbf{0}$ as zero element and M as the identity, but it is not associative. The verification of idempotency and commutativity is routine. Figure 6 shows two faults M^a

and M^b. Letting $M^1 = M^a$, and $M^2 = M^3 = M^b$, we find that the transition from state 10 under input w_1^i results in $p_j^{(1\diamond 2)\diamond 3} = 1$ and $p_j^{1\diamond(2\diamond 3)} = 0$.

Since our main interest is in the Thatte-Abraham faults, we add a restriction to obtain a class of faults in which $\diamond$ is associative.

A RAM delta fault $M' = (Q', \delta')$ is *change-attempt-activated* if for all $q \in Q'$, $i \in \{1, \ldots, n\}$, and $x \in X_i$,

$$\delta_i(q, x) = q_i \Longrightarrow \delta'(q, x) = q.$$

In such a fault, no change of state is possible, unless the state of the addressed cell is being changed in the fault-free RAM. Let $\mathcal{A}$ be the set of change-attempt-activated faults.

Let $\mathcal{C}$ be the set of delta faults satisfying, for all $q \in Q'$, $i \in \{1, \ldots, n\}$, and $x \in X_i$,

$$\delta_i(q, x) \neq q_i = \delta_i'(q, x) \Longrightarrow \delta'(q, x) = q.$$

Let $\mathcal{S} = \mathcal{A} \cap \mathcal{C}$ be the set of *change-success-activated* faults. One verifies that all the faults in $\mathcal{T}$ are in $\mathcal{S}$.

In the theorem below, $(\mathcal{T}^\diamond, \diamond)$ denotes the semilattice generated by $\mathcal{T}$ under the operation $\circ$.

Theorem 3. *The algebra $(\mathcal{S} \cap \mathcal{P}, \diamond)$ is a semilattice with* $\mathbf{0}$ *as zero element and M as the identity, and $\diamond$ satisfies ρ_1–ρ_5. The algebra $(\mathcal{T}^\diamond, \diamond)$ is a proper subsemilattice of $(\mathcal{S} \cap \mathcal{P}, \diamond)$. Furthermore, equations ρ_1–ρ_4 form a minimal set of defining relations for $\mathcal{T}^\diamond$.*

Proof. The diamond operation was introduced in [3], as an composition operation on delta faults. However, we later discovered that this operation was ill-defined for some delta faults; consequently, several results in [3] are incorrect as stated. The subproduct condition should be added, as it is done in the present paper, to obtain correct statements. With this added condition, the proofs of [3] apply for the first two claims of the theorem. The proof of the third claim is similar to the proof of the corresponding statement in Theorem 2.

6 Characterization of $\mathcal{T}^\circ$ and $\mathcal{T}^\diamond$

Roughly speaking, a RAM delta fault involving a set C of cells is pattern-sensitive if it occurs only for some values of the cells that are not in C. More precisely, A delta fault $M' = (Q', \delta')$ is *pattern-sensitive* if one of the following conditions holds:
(1) There exist $i \in \{1, \ldots, n\}$, $x \in X_i$, and $q, q' \in Q'$ such that $q_i' = q_i$ and $\delta_i'(q, x) \neq \delta_i'(q', x)$.
(2) There exist $i, j \in \{1, \ldots, n\}$, $x \in X_i$, and $q, q' \in Q'$ such that $q_i' = q_i$, $q_j' = q_j$, and $\delta_j'(q, x) \neq \delta_j'(q', x)$.
Otherwise, it is *pattern-insensitive.* Let $\mathcal{I}$ be the set of pattern-insensitive faults.

Lemma 1. *The sets $\mathcal{A}$, $\mathcal{S}$ and $\mathcal{I}$ are closed under $\circ$ and $\diamond$.*

Lemma 2. *Let $M^1, M^2 \in \Delta$ with $M^1 = (Q^1, \delta^1)$ and $M^2 = (Q^2, \delta^2)$, and let $M = (Q, \delta)$ be the corresponding fault-free automaton.*

1. *For all $q \in Q^{1\circ 2}$ and all $x \in X$, if $\delta^1(q,x) = \delta(q,x)$ and $\delta^2(q,x) \in Q^{1\circ 2}$ then $\delta^{1\circ 2}(q,x) = \delta^2(q,x)$.*
2. *If $M^2 \in \mathcal{C}$ then, for all $q \in Q^{1\diamond 2}$ and all $x \in X$, if $\delta^1(q,x) = \delta(q,x)$ and $\delta^2(q,x) \in Q^{1\diamond 2}$ then $\delta^{1\diamond 2}(q,x) = \delta^2(q,x)$.*

Theorem 4. *(a) A delta fault is change-attempt-activated, pattern-insensitive, and a subproduct fault if and only if it is equivalent to a $\circ$-composition of Thatte-Abraham faults, that is, $\mathcal{T}^\circ = \mathcal{A} \cap \mathcal{I} \cap \mathcal{P}$. (b) A delta fault is change-success-activated, pattern-insensitive, and a subproduct fault if and only if it is equivalent to a $\diamond$-composition of Thatte-Abraham faults, that is, $\mathcal{T}^\diamond = \mathcal{C} \cap \mathcal{T}^\circ$.*

Proof. By the definition of $\mathcal{T}$ one has $\mathcal{T} \subseteq \mathcal{A} \cap \mathcal{I} \cap \mathcal{P} \cap \mathcal{C}$. By Lemma 1 and Theorems 2 and 3, $\mathcal{A} \cap \mathcal{I} \cap \mathcal{P}$ is closed under $\circ$ and $\mathcal{A} \cap \mathcal{I} \cap \mathcal{P} \cap \mathcal{C}$ is closed under $\diamond$. Therefore, $\mathcal{T}^\circ \subseteq \mathcal{A} \cap \mathcal{I} \cap \mathcal{P}$ and $\mathcal{T}^\diamond \subseteq \mathcal{A} \cap \mathcal{I} \cap \mathcal{P} \cap \mathcal{C}$.

We first show that $\mathcal{T}^\circ \supseteq \mathcal{A} \cap \mathcal{I} \cap \mathcal{P}$. Consider $M' = (Q', \delta') \in \mathcal{A} \cap \mathcal{I} \cap \mathcal{P}$. As $M' \in \mathcal{A}$, errors can arise only from writing a new value. Let $\sigma(M')$ be the number of state set components Q'_j which are singleton sets; let $\tau(M')$ be the number of pairs (i,a) with $i \in \{1, \ldots, n\}$ and $a \in \{0,1\}$ such that, for some $q \in Q'$ with $q_i = \overline{a}$, one has $\delta'_i(q, w^i_a) \neq a$; let $\gamma(M')$ be the number of quadruples (i,a,j,b) with $i,j \in \{1,\ldots,n\}$, $i \neq j$, and $a,b \in \{0,1\}$ such that, for some $q \in Q'$ with $q_i = \overline{a}$ and $q_j = \overline{b}$, one has $\delta'_j(q, w^i_a) = b$.

We show that M' can be represented as $M' = S \circ T \circ C \circ M$ where S is a $\circ$-composition of stuck-at faults, T is a $\circ$-composition of transition faults, and C is a $\circ$-composition of coupling faults. In all three cases, the $\circ$-composition of zero faults is M.

We first determine the component S. By splitting S off, we may then assume that $\sigma(M') = 0$. We then determine the component T. By splitting T off, we may assume that also $\tau(M') = 0$. Third, we split C off, leaving a semiautomaton M' with $\sigma(M') = \tau(M') = \gamma(M') = 0$. One verifies that this implies $M' = M$, given that $M' \in \mathcal{A} \cap \mathcal{I} \cap \mathcal{P}$. We proceed by induction, first on $\sigma(M')$, then on $\tau(M')$, and then on $\gamma(M')$. If $M' \in \mathcal{T}$ or $M' = \mathbf{0}$, nothing needs to be proved. Assume, therefore, that $M' \notin \mathcal{T}$ and $M' \neq \mathbf{0}$.

First suppose that $\sigma(M') > 0$. Then $Q'_j = \{a\}$ for some j and some a. Let $M^1 = M^{j=a}$ and let $M^2 = (Q^2, \delta^2)$ be defined as follows. Let

$$Q^2 = Q'_1 \times \cdots \times Q'_{j-1} \times \{0,1\} \times Q'_{j+1} \times \cdots \times Q'_n.$$

For $q \in Q^2$, let $q' = (q_1, \ldots, q_{j-1}, a, q_{j+1}, \ldots, q_n)$. For all inputs x and for all $i \neq j$, $\delta^2_i(q,x) = \delta'_i(q',x)$. For component j, $\delta^2_j(q,x)$ is $\overline{q}_j$, if $x = w^j_{\overline{q}_j}$, and q_j otherwise.

We verify that $M^1 \circ M^2 = M'$. Clearly, $Q' = Q^1 \cap Q^2$. Consider $q \in Q'$ and $x \in X$. If $x \notin X_j$ then $\delta^1(q,x) = \delta(q,x)$ and $\delta^2(q,x) \in Q'$; hence $\delta^{1\circ 2}(q,x) = \delta^2(q,x)$ by Lemma 2 and $\delta^2(q,x) = \delta'(q,x)$. Now consider $x \in X_j$. If $x = w_a^j$ then $\delta'(q,x) = q$ as $M' \in \mathcal{A}$ and $\delta^{1\circ 2}(q,x) = q = \delta'(q,x)$. Now let $x = w_{\overline{a}}^j$. Then, by the definition of $\circ$, $\delta^{1\circ 2}(q,x) = \delta'(q,x)$.

By construction, $M^2 \in \mathcal{A} \cap \mathcal{I} \cap \mathcal{P}$ and $\sigma(M^2) < \sigma(M')$. Hence, by induction, M^2 can be written in the form $M^2 = M'' \circ M'''$ where M'' is a $\circ$-composition of stuck-at faults and $M''' \in \mathcal{A} \cap \mathcal{I} \cap \mathcal{P}$ with $\sigma(M''') = 0$. Then $(M^1 \circ M'') \circ M'''$ is the required representation of M' with $S = M^1 \circ M''$.

Now we consider the case of $\sigma(M') = 0$. Then $Q' = Q = \{0,1\}^n$. Suppose that $\tau(M') > 0$. Then there is a cell i such w_a^i fails, that is, there is a state q with $q_i = \overline{a}$ such that $\delta_i'(q, w_a^i) = \overline{a}$. Let $M^1 = M^{i\neg a}$ and let $M^2 = (Q, \delta^2)$ be defined as

$$\delta^2(q,x) = \begin{cases} \delta'(q,x), & \text{if } x \neq w_a^i, \\ (\delta_1'(q,x), \ldots, \delta_{i-1}'(q,x), a, \delta_{i+1}'(q,x), \ldots, \delta_n'(q,x)), & \text{if } x = w_a^i, \end{cases}$$

for $q \in Q$ and $x \in X$. Using the fact that $M' \in \mathcal{A} \cap \mathcal{I}$ one verifies that $M' = M^1 \circ M^2$. By construction, $M^2 \in \mathcal{A} \cap \mathcal{I} \cap \mathcal{P}$, $\sigma(M^2) = 0$, and $\tau(M^2) < \tau(M')$ as $M' \in \mathcal{I}$. Hence, by induction, M^2 can be written in the form $M^2 = M'' \circ M'''$ where M'' is a $\circ$-composition of transition faults and $M''' \in \mathcal{A} \cap \mathcal{I} \cap \mathcal{P}$ with $\sigma(M''') = 0 = \tau(M''')$. Then $(M^1 \circ M'') \circ M'''$ is the required representation of M' with $T = M^1 \circ M''$.

Finally, assume that $\sigma(M') = \tau(M') = 0$ and $\gamma(M') > 0$. Then there are distinct cells i and j and values $a, b \in \{0,1\}$ such that for some state $q \in Q$ with $q_i = \overline{a}$ and $q_j = \overline{b}$ one has $\delta_j'(q, w_a^i) = b$. Let $M^1 = M^{ia \Rightarrow jb}$ and let $M^2 = (Q, \delta^2)$ be given by

$$\delta_k^2(q,x) = \begin{cases} b, & \text{if } x = w_a^i,\ q_i = \overline{a},\ q_j = \overline{b}, \text{ and } k = j, \\ \delta_k'(q,x), & \text{otherwise}, \end{cases}$$

for $q \in Q$, $k = 1, \ldots, n$, and $x \in X$. We prove that $M' = M^1 \circ M^2$.

Clearly, $Q' = Q^{1\circ 2} = Q$. Consider $q \in Q$ and $x \in X$. If $x \neq w_a^i$ or $q_i \neq \overline{a}$ or $q_j \neq \overline{b}$ then $\delta^1(q,x) = \delta(q,x)$; hence $\delta^{1\circ 2}(q,x) = \delta^2(q,x) = \delta'(q,x)$.

Now suppose that $x = w_a^i$, $q_i = \overline{a}$ and $q_j = \overline{b}$. Then

$$\delta_k^{1\circ 2}(q,x) = \begin{cases} \delta_k'(q,x), & \text{if } k \neq j, \\ b, & \text{if } k = j, \end{cases}$$

for $k = 1, \ldots, n$. As $M' \in \mathcal{I}$, this is equal to $\delta_k'(q,x)$.

By construction, $M^2 \in \mathcal{A} \cap \mathcal{I} \cap \mathcal{P}$, $\sigma(M^2) = 0 = \tau(M')$, and $\gamma(M^2) < \gamma(M')$. Hence, by induction, M^2 can be written in the form of a $\circ$-composition of coupling faults and, therefore, also M' can be represented in this way. This completes the proof of statement (a).

The proof of the inclusion $\mathcal{T}^\diamond \supseteq \mathcal{A} \cap \mathcal{I} \cap \mathcal{P} \cap \mathcal{C}$ is analogous to the one for $\circ$ except that the faults to be considered have to be also in $\mathcal{C}$ at every induction step.

Proposition 2. *There is a homomorphism of* $\mathcal{T}^\circ$ *onto* $\mathcal{T}^\diamond$ *such that its restriction to* $\mathcal{T}^\diamond$ *is the identity mapping. For* $M^1, M^2 \in \mathcal{T}^\diamond$, *if* $M^1 \circ M^2 \in \mathcal{C}$, *then* $M^1 \circ M^2 = M^1 \diamond M^2$.

Proof. The two semilattices have equations ρ_1–ρ_3 in common.

Acknowledgments

This work was supported by the Natural Sciences and Engineering Research Council of Canada, under Grants OGP0000871 and OGP0000243. The authors thank Jacques Sakarovitch for his suggestions regarding the construction of representatives in Section 2, and Gerard Lallement for pointing out reference [4].

References

1. J. A. Brzozowski and B. F. Cockburn: Detection of Coupling Faults in RAMs. *J. of Electronic Testing: Theory and Applications* **1** (1990), 151–162.
2. J. A. Brzozowski, H. Jürgensen: Component Automata and RAM Faults. *Proc. 2nd International Colloquium on Words, Languages, and Combinatorics, Kyoto, Japan, 1992*, edited by M. Ito and H. Jürgensen. World Scientific, Singapore, 1994, 49–67.
3. J. A. Brzozowski, H. Jürgensen: An Algebra of Multiple Faults in RAMs. *J. of Electronic Testing: Theory and Applications* **8** (1996), 129–142.
4. T. Evans: Some Connections between Residual Finiteness, Finite Embeddability and the Word Problem. *J. London Math. Soc. (2)* **1** (1969), 399–403.
5. F. Gésceg: *Products of Automata*, Springer-Verlag, Berlin, 1986.
6. A. J. van de Goor: *Testing Semiconductor Memories, Theory and Practice*, Wiley, Chichester, England, 1991.
7. S. M. Thatte, J. A. Abraham: Testing of Semiconductor Random-Access Memories. *Digest of Papers, 7th Int. Conf. on Fault-Tolerant Computing*, Los Angeles, June 28–30, 1977, 81–87.