Involuted Semilattices and Uncertainty in Ternary Algebras *

J. A. Brzozowski
Department of Computer Science
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1
brzozo@uwaterloo.ca

April 3, 2002

Abstract

An involuted semilattice $\langle S, \vee, ^- \rangle$ is a semilattice $\langle S, \vee \rangle$ with an involution $^- : S \to S$, *i.e.*, $\langle S, \vee, ^- \rangle$ satisfies $\overline{\overline{a}} = a$, and $\overline{a \vee b} = \overline{a} \vee \overline{b}$. In this paper we study the properties of such semilattices. In particular, we characterize free involuted semilattices in terms of ordered pairs of subsets of a set. An involuted semilattice $\langle S, \vee, ^-, 1 \rangle$ with greatest element 1 is said to be complemented if it satisfies $a \vee \overline{a} = 1$. We also characterize free complemented semilattices. We next show that complemented semilattices are related to ternary algebras. A ternary algebra $\langle T, +, *, ^-, 0, \phi, 1 \rangle$ is a de Morgan algebra with a third constant ϕ satisfying $\phi = \overline{\phi}$, and $(a + \overline{a}) + \phi = a + \overline{a}$. If we define a third binary operation $\vee$ on T as $a \vee b = a * b + (a + b) * \phi$, then $\langle T, \vee, ^-, \phi \rangle$ is a complemented semilattice.

1 Semilattices

A *semilattice* [7, 8] $\langle S, \vee \rangle$ is a nonempty set S together with a binary operation $\vee$, such that equations S1–S3 below are satisfied for all $a, b, c \in S$.

*This research was supported by Grant No. OGP0000871 from the Natural Sciences and Engineering Research Council of Canada.

$$
\begin{array}{ll}
\text{S1.} & a \vee a = a \\
\text{S2.} & a \vee b = b \vee a \\
\text{S3.} & a \vee (b \vee c) = (a \vee b) \vee c
\end{array}
$$

We define a partial order on S as follows:

$$a \preceq b \text{ iff } a \vee b = b,$$

and also use $\succeq$, $\prec$, and $\succ$ in the usual sense.

In most of this paper we are concerned with finite semilattices. Every finite semilattice has a unique greatest element, the least upper bound, *lub*, of S, which we call 1. Thus,

$$1 = \mathit{lub}\ S = \bigvee_{s \in S} s.$$

It follows that every finite semilattice also satisfies S4 below; in this case we denote it by $\langle S, \vee, 1\rangle$, to identify the constant 1.

$$\text{S4.} \quad a \vee 1 = 1.$$

For any element a of a semilattice $\langle S, \vee, 1\rangle$, we define

$$a \vee S = \{a \vee s \mid s \in S\} = \{t \in S \mid t \succeq a\}.$$

Proposition 1 *Let $\langle S, \vee, 1\rangle$ be a finite semilattice. Then, for any $a \in S$, $\langle a \vee S, \vee, 1\rangle$ is a lattice with a as zero.*

Proof: Clearly, $a \vee S$ is a subsemilattice of S. We need to show that every two elements b and c in S such that $b, c \succeq a$ have a greatest lower bound (*glb*). Let

$$Q = \{d \mid d \preceq b \text{ and } d \preceq c\}$$

be the set of all lower bounds of the set $\{b, c\}$. Since $a \in Q$, Q is nonempty. Then

$$\mathit{glb}\{b, c\} = \bigvee_{e \in Q} e.$$

Consequently, $a \vee S$ is a lattice, and, by definition, a is its smallest element, and hence the zero of the lattice. □

If $\mathit{glb}\{a, b\}$ exists, we denote it by $a \wedge b$.

2 Involuted semilattices

We use the terminology introduced by Bredikhin (see, for example, [2, 3]), although our notation is different. We call $\langle S, \vee, ^{-} \rangle$ an *involuted semigroup*[1] if $\langle S, \vee \rangle$ is a semigroup, and $^{-} : S \to S$ is a unary operation of *involution*, *i.e.,* satisfies the properties

$$\text{S5.} \quad \overline{\overline{a}} = a,$$

and $\overline{a \vee b} = \overline{b} \vee \overline{a}$. If we are dealing with semilattices, the commutative law S2 holds, and the latter equation takes the form

$$\text{S6.} \quad \overline{a \vee b} = \overline{a} \vee \overline{b}.$$

Thus, by an *involuted semilattice* $\langle S, \vee, ^{-}, 1 \rangle$ we mean an algebra satisfiying S1–S6.[2] We refer to the unary operation $^{-}$ as *quasi-complementation*.[3]

Example 1 *Let A be a set and let S be the set of all binary relations on A. Then $\langle S, \cup, ^{-1}, A \times A \rangle$, where $^{-1}$ denotes the converse of a relation, is an involuted semilattice.* $\square$

Example 2 *Let Σ be a finite alphabet and Σ^*, the free monoid generated by Σ. Let w^{-1} be the reverse of w for any $w \in \Sigma^*$. If $L \subseteq \Sigma^*$, let $L^{-1} = \{w^{-1} \mid w \in L\}$. Let S be the set of all subsets of Σ^*. Then $\langle S, \cup, ^{-1}, \Sigma^* \rangle$ is an involuted semilattice.* $\square$

In an involuted semilattice S, if $a = \overline{a}$, we say that a is *self-complementary*. The set of all self-complementary elements of S will be denoted by $C(S)$.

If $a \prec b$ and there is no c such that $a \prec c \prec b$, we say that b *covers* a, or a *is covered by* b, and denote this by $a \lessdot b$.

Let $\langle S, \vee \rangle$ and $\langle T, \circ \rangle$ be semilattices, and $h : S \to T$ a semilattice homomorphism, *i.e.,* a mapping from S to T satisfying:

$$h(a \vee b) = h(a) \circ h(b),$$

[1] Bredikhin studied involuted semigroups [2] with an *external* partial order, or ordered by an *external* semilattice [3], whereas we study involuted semigroups which *are* semilattices, *i.e.,* are idempotent and commutative.

[2] In this paper semilattices always have the greatest element 1.

[3] This is a different concept than pseudo-complementation [7].

for all $a, b \in S$. Then, if $a \vee b = b$ in S, we have $h(a) \circ h(b) = h(b)$ in T; thus, semilattice homomorphisms preserve the semilattice order. Since the involution $^-$ is a homomorphism from S to S, in fact, an automorphism of S, we have

$$a \preceq b \text{ implies } \overline{a} \preceq \overline{b}.$$

It also follows from the definition of $\preceq$ that

$$a \preceq c \text{ and } b \preceq d \text{ implies } (a \vee b) \preceq (c \vee d).$$

Thus, both operations are monotonic.

Proposition 2 *In an involuted semilattice $\langle S, \vee, ^-, 1\rangle$ we have:*

1. *$a \preceq b$ iff $\overline{a} \preceq \overline{b}$.*
2. *$a \lhd b$ iff $\overline{a} \lhd \overline{b}$.*
3. *a is self-complementary iff $a = a \vee \overline{a}$.*
4. *1 is self-complementary, i.e., $\overline{1} = 1$.*
5. *$\langle C(S), \vee, ^-, 1\rangle$ is a sub-involuted-semilattice of S.*
6. *if $a \neq \overline{a}$, then a and $\overline{a}$ are incomparable with respect to $\preceq$.*
7. *For every chain $a_n \lhd \ldots \lhd a_1 \lhd 1$ there is a corresponding chain $\overline{a}_n \lhd \ldots \lhd \overline{a}_1 \lhd 1$.*

Proof: First, $a \preceq b$ iff $a \vee b = b$ iff $\overline{a \vee b} = \overline{b}$ iff $\overline{a} \vee \overline{b} = \overline{b}$ (by S6) iff $\overline{a} \preceq \overline{b}$.

The second claim follows immediately from 1).

Third, if $a = a \vee \overline{a}$, then $\overline{a} = \overline{a \vee \overline{a}} = \overline{a} \vee \overline{\overline{a}} = \overline{a} \vee a = a \vee \overline{a} = a$, by S6, S5, and S2. Conversely, if $a = \overline{a}$, then $a = a \vee a = a \vee \overline{a}$.

Fourth, $\overline{1} = \overline{a \vee 1} = \overline{a} \vee \overline{1}$ by S4 and S6. Thus, $\overline{1}$ is the greatest element. Since the greatest element is unique, we have $\overline{1} = 1$.

Fifth, if $a, b \in C(S)$, then $a = \overline{a}$ and $b = \overline{b}$. Hence, $a \vee b = \overline{a} \vee \overline{b} = \overline{a \vee b}$.

Sixth, suppose that $a \neq \overline{a}$ and $a \succeq \overline{a}$, *i.e.,* $a \vee \overline{a} = a$. Then $\overline{a} = \overline{a \vee \overline{a}} = a \vee \overline{a} = a$, contradicting that $a \neq \overline{a}$. A similar argument holds if $a \preceq \overline{a}$.

The last claim follows from 2) and 4). □

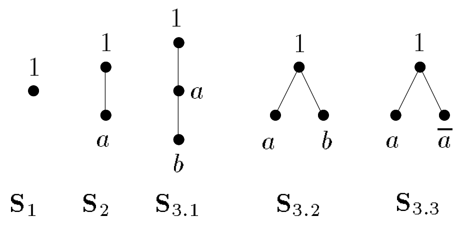

Figure 1: Involuted semilattices with ≤ 3 elements.

Example 3 *Figure 1 shows all involuted semilattices with ≤ 3 elements. We use the convention that, if a is an element of a Hasse diagram and there is no element in the diagram labeled $\overline{a}$, then $a = \overline{a}$.*

One also verifies that there are eight involuted semilattices with four elements; see Fig. 2. In the first five, all elements are self-complementary. For such semilattices S5 and S6 hold trivially; hence, there are as many such involuted semilattices as there are semilattices. □

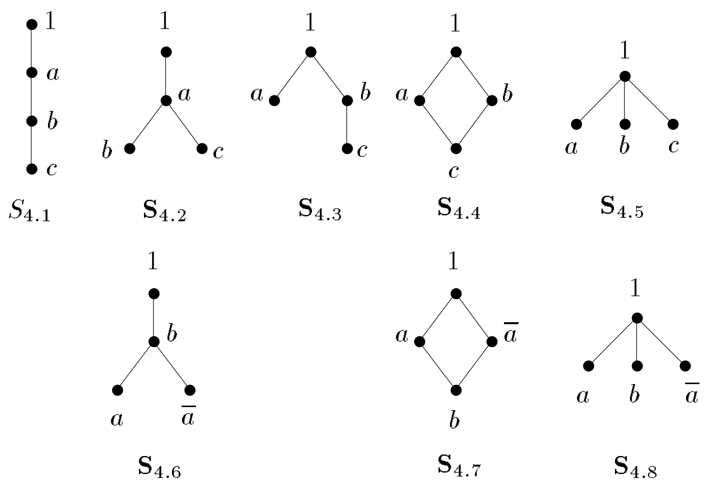

Figure 2: Involuted semilattices with 4 elements.

If $\langle S, \vee, ^-, 1\rangle$ is an involuted semilattice and $a \in S$, let

$$S(a) = (a \vee S) \cup (\overline{a} \vee S).$$

Thus, $S(a)$ is the set of all elements of S that are above a or above $\overline{a}$, or both. Note that, if S is finite, by Proposition 1, both $a \vee S$ and $\overline{a} \vee S$ are lattices.

Let $K(a) = (a \vee S) \cap (\overline{a} \vee S)$.

Proposition 3 *The following hold for an involuted semilattice $\langle S, \vee, ^{-}, 1\rangle$ and $a \in S$:*

1. *$b \in K(a)$ iff $b = b \vee a \vee \overline{a}$; thus, $K(a) = S(a \vee \overline{a})$.*
2. *$\overline{a} \vee S = \{\overline{b} \mid b \in a \vee S\}$.*
3. *$K(a) \supseteq C(S(a))$, where $C(S(a))$ is the set of self-complementary elements of $S(a)$.*
4. *The mapping $^{-} : a \vee S \to \overline{a} \vee S$ is a semilattice isomorphism.*
5. *If S is finite or a lattice, the mapping $^{-} : a \vee S \to \overline{a} \vee S$ is a lattice isomorphism.*
6. *$\langle S(a), \vee, ^{-}, 1\rangle$ and $\langle K(a), \vee, ^{-}, 1\rangle$ are sub-involuted-semilattices of S.*

Proof: First, if $b \in K(a)$, we have $b \vee a = b$ and $b \vee \overline{a} = b$. Hence, $b = b \vee a \vee \overline{a}$. Conversely, $b = b \vee a \vee \overline{a}$ implies $b = b \vee a$ and $b = b \vee \overline{a}$. It now follows that $K(a) = S(a \vee \overline{a})$.

Second, $b \in \overline{a} \vee S$ iff $b = \overline{a} \vee b$ iff $\overline{b} = a \vee \overline{b}$ iff $\overline{b} \in a \vee S$.

Third, if $b \in S(a)$, and $b = \overline{b}$, then $b \in K(a)$, by 2).

The fourth and fifth claims follow from the fact that $^{-}$ is an order isomorphism, as shown in Proposition 2, 1).

The sixth claim is easily verified. □

Example 4 *Let $[n] = \{1, \ldots, n\}$, and let P_n be the set of all ordered pairs (A, A') of subsets of $[n]$, where A and A' are not both empty, i.e.,*

$$P_n = \{(A, A') \mid A, A' \subseteq [n], A \cup A' \neq \emptyset\}.$$

Define operation $\vee$ on P_n as follows.

$$(A, A') \vee (B, B') = (A \cup B, A' \cup B').$$

Furthermore, let

$$\overline{(A, A')} = (A', A),$$

and $1_P = ([n],[n])$. *One verifies that* $\langle P_n, \vee, ^-, 1_P\rangle$ *is an involuted semilattice.*

In particular, consider $S = P_2$, *and* $a = (1, \emptyset)$. *We show* $S(a)$ *in Fig. 3, where we denote* $\{1\}$ *by* 1, $\{1,2\}$, *by* 12, *etc., to simplify the notation. Here,* $a \vee S = \{(1,\emptyset), (12,\emptyset), (1,2), (1,1), (12,1), (12,12)\}$, $C(S(a)) = \{(1,1), (12,12)\}$, *and* $K(a) = \{(1,1), (12,1), (1,12), (12,12)\}$. ☐

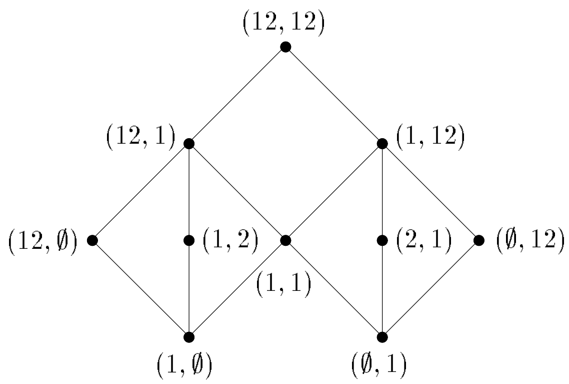

Figure 3: Illustrating $S(a)$ for P_2.

3 Free involuted semilattices

Let Q be any set, and define

$$P(Q) = \{(A, A') \mid A, A' \subseteq Q, A, A' \text{ are finite, and } A \cup A' \neq \emptyset\}.$$

Define the following operations on $P(Q)$:

$$(A, A') \vee (B, B') = (A \cup B, A' \cup B'),$$

and

$$\overline{(A, A')} = (A', A).$$

Let 1_P be an element that is not in $P(Q)$, and define $1_P \vee 1_P = 1_P \vee p = p \vee 1_P = 1_P$ for all $p \in P(Q)$, and $\overline{1_P} = 1_P$. Then

$$\langle P(Q) \cup \{1_P\}, \vee, ^-, 1_P\rangle$$

is an involuted semilattice.

Let $g_q = (\{q\}, \emptyset)$, for every $q \in Q$, and let $G(Q) = \{g_q \mid q \in Q\}$. Then $P(Q)$ is generated by $G(Q)$. In fact, any element (A, A') of $P(Q)$ can be expressed as

$$(A, A') = \bigvee_{q \in A} (\{q\}, \emptyset) \vee \overline{\bigvee_{q \in A'} (\{q\}, \emptyset)} = \bigvee_{q \in A} g_q \vee \overline{\bigvee_{q \in A'} g_q}.$$

Let $\langle S, \vee, ^-, 1_S \rangle$ and $\langle T, \circ, \neg, 1_T \rangle$ be involuted semilattices, and $h : S \to T$ a mapping from S to T. Then h is an involuted-semilattice homomorphism if it preserves the operations and the constant, *i.e.,* if $h(a \vee b) = h(a) \circ h(b)$, $h(\overline{a}) = \neg(h(a))$, and $h(1_S) = 1_T$.

Theorem 1 *$\langle P(Q) \cup \{1_P\}, \vee, ^-, 1_P \rangle$ is freely generated by $G(Q)$ in the class of involuted semilattices.*

Proof: Let $\langle S, \vee, ^-, 1 \rangle$ be an involuted semilattice and let $\mu : G(Q) \to S$ be any mapping. We extend μ to a mapping from $P(Q) \cup \{1_P\}$ to S as follows. If (A, A') is any element of $P(Q)$, define

$$\mu((A, A')) = \bigvee_{q \in A} \mu(g_q) \vee \overline{\bigvee_{q \in A'} \mu(g_q)}.$$

Also, let $\mu(1_P) = 1$.

We need to verify that μ is a semilattice homomorphism. We have

$$\mu(\overline{(A, A')}) = \mu((A', A)) = \bigvee_{q \in A'} \mu(g_q) \vee \overline{\bigvee_{q \in A} \mu(g_q)}.$$

On the other hand,

$$\overline{\mu((A, A'))} = \overline{\bigvee_{q \in A} \mu(g_q) \vee \overline{\bigvee_{q \in A'} \mu(g_q)}} = \overline{\bigvee_{q \in A} \mu(g_q)} \vee \bigvee_{q \in A'} \mu(g_q) = \mu(\overline{(A, A')}).$$

Also $\mu(\overline{1_P}) = \mu(1_P) = 1 = \overline{1} = \overline{\mu(1_P)}$.

For the binary operation,

$$\mu((A, A') \vee (B, B')) = \mu(A \cup B, A' \cup B') = \bigvee_{q \in A \cup B} \mu(g_q) \vee \overline{\bigvee_{q \in A' \cup B'} \mu(g_q)} =$$

$$\bigvee_{q \in A} \mu(g_q) \vee \bigvee_{q \in B} \mu(g_q) \vee \overline{\bigvee_{q \in A'} \mu(g_q)} \vee \overline{\bigvee_{q \in B'} \mu(g_q)} = \mu((A, A')) \vee \mu((B, B')).$$

Finally, $\mu(1_P \vee p) = \mu(p \vee 1_P) = \mu(1_P) = 1 = \mu(1_P) \vee \mu(p) = \mu(p) \vee \mu(1_P)$, as required. Since the operations and the constant are preserved, μ is a homomorphism. Hence, our claim holds. $\square$

One verifies that, if Q has n elements, there are $2^{2n} - 1$ elements in the free involuted semilattice $P(Q)$.

4 Complemented semilattices

A *complemented semilattice*[4] is an involuted semilattice $\langle S, \vee, ^-, 1\rangle$ satisfying

S7. $a \vee \overline{a} = 1$.

Proposition 4 *Complemented semilattices have the following properties:*

1. *An involuted semilattice is complemented iff* 1 *is the only element that is self-complementary, i.e., if* $a \neq 1$*, then* $a \neq \overline{a}$*.*

2. *All chains* $a_n \lhd \ldots a_1 \lhd 1$ *and* $\overline{a}_n \lhd \ldots \overline{a}_1 \lhd 1$ *are disjoint except for* 1*.*

3. *If* $a \neq 1$*, there is no element* c *such that* $c \preceq a$ *and* $c \preceq \overline{a}$*. Consequently,* $a \wedge \overline{a}$ *does not exist.*

4. *A finite complemented semilattice has an odd number of elements.*

5. *In a finite complemented semilattice there is an even number of elements* a *such that* $a \lhd 1$*.*

Proof: First, suppose $a = \overline{a}$ for some $a \in S$. Then $a = a \vee a = a \vee \overline{a} = 1$. Conversely, suppose $a = \overline{a}$ implies $a = 1$ for all $a \in S$. Since $a \vee \overline{a} = \overline{a \vee \overline{a}}$, we must have $a \vee \overline{a} = 1$.

The second claim follows, since only 1 is self-complementary.

Third, if $c \preceq a$ and $c \preceq \overline{a}$, then also $\overline{c} \preceq a$ and $\overline{c} \preceq \overline{a}$. Hence, $c \vee \overline{c} \preceq a \vee a = a$ and $c \vee \overline{c} \preceq \overline{a}$. But $c \vee \overline{c} = 1$ in a complemented semilattice. Hence $a = 1$, which is a contradiction.

Fourth, S is the union of sets of the form $\{a, \overline{a}\}$, all of which have two elements, except in the case when $a = 1$.

For the fifth claim, suppose there is an odd number of elements covered by 1. Then there must be at least one a such that $a \lhd 1$, but $\overline{a} \lhd 1$ is false. But this contradicts Proposition 2 (2) that $\overline{a} \lhd \overline{1}$, *i.e.,* $\overline{a} \lhd 1$. $\square$

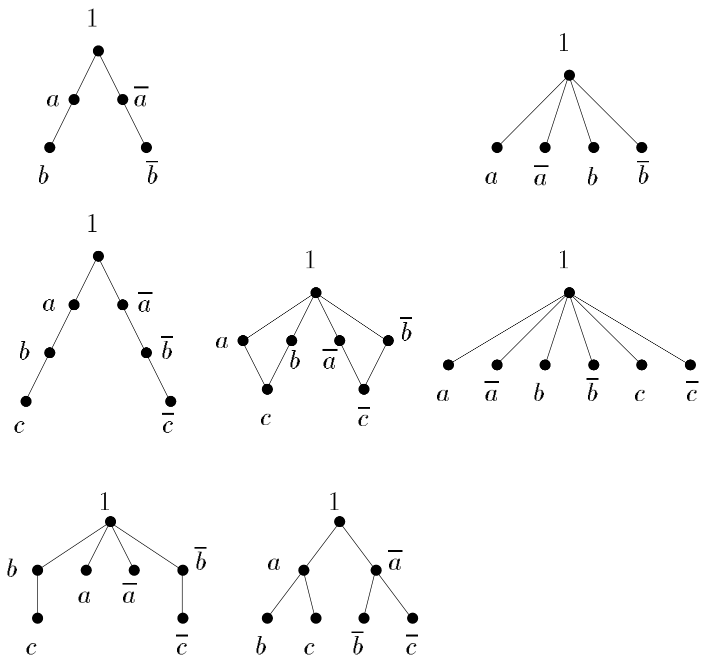

Figure 4: Complemented semilattices with 5 and 7 elements.

Example 5 *There is one complemented semilattice with one element, namely* $\mathbf{S}_1$*, and one with three elements, namely* $\mathbf{S}_{3.3}$*, as shown in Fig. 1. Complemented semilattices with five and seven elements are shown in Fig. 4.* □

For a complemented semilattice $\langle S, \vee, ^-, 1\rangle$ and $a \in S$, we have:

$$K(a) = S(a \vee \overline{a}) = \{1\}.$$

Example 6 *Return to Example 4. Let* $\sim$ *be the equivalence relation on* $P_2(a)$ *from Example 4 which puts into one class* I *all the pairs* (A, A') *such that* $A \cap A' \neq \emptyset$*, and treats all other pairs as singleton classes. Consider* $\langle E_2, \vee, ^-, I\rangle$*, where* E_2 *is the set of equivalence classes with respect to* $\sim$*. We obtain the semilattice of Fig. 5, where the singleton classes are identified with their elements.* □

[4]This notion still differs from that of a pseudo-complemented semilattice [7].

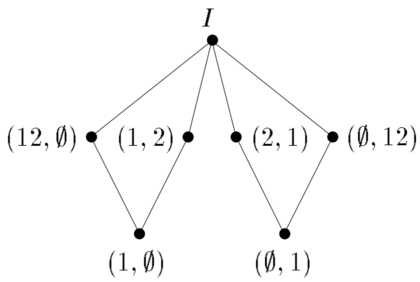

Figure 5: Illustrating $S(a)$ for E_2.

5 Free complemented semilattices

Return now to the involuted semilattice $\langle P(Q) \cup \{1_P\}, \vee, ^-, 1_P \rangle$ of Section 3. Define an equivalence relation $\sim$ on $P(Q) \cup \{1_P\}$ as follows:

$$\text{if } A \cap A' = \emptyset, \text{ then } (A, A') \sim (B, B') \text{ iff } A = B, \text{ and } A' = B',$$

$$\text{if } A \cap A' \neq \emptyset, \text{ then } (A, A') \sim (B, B') \text{ iff } B \cap B' \neq \emptyset \text{ and } (A, A') \sim 1_P,$$

$$\text{also } 1_P \sim 1_P.$$

Let $[(A, A')]$ denote the equivalence class of (A, A') with respect to $\sim$. There are two types of equivalence classes. For each pair (A, A') of subsets of Q,

$$[(A, A')] = \{(A, A')\} \text{ if } A \cap A' = \emptyset,$$

and

$$I = \{(A, A') \mid A \cap A' \neq \emptyset\} \cup \{1_P\}.$$

Let $E(Q)$ denote the set of all equivalence classes of $\sim$. The operations $\vee$ and $^-$ are defined on $E(Q)$ in the usual way: for $p, q \in P(Q)$, $[p] \vee [q] = [p \vee q]$, and $\overline{[p]} = [\overline{p}]$. One verifies that the operations $\vee$ and $^-$ are well defined on $E(Q)$. Clearly, $[\overline{p}]$ is unique if $[p]$ is a singleton. Otherwise, p must be of the form (A, A'), where $A \cap A' \neq \emptyset$, or $p = 1_P$. Any q equivalent to p also has the nonempty intersection property or is 1_P, and the same is true of $\overline{q}$. Hence, $[\overline{p}] = [\overline{q}] = I$. For $\vee$, suppose at least one of p, q is in I. Then so is $p \vee q$, and $[p \vee q]$ is uniquely I. If neither p nor q is in I, then $[p]$ and $[q]$ are singletons, and $[p \vee q]$ is uniquely defined.

It follows that $\langle E(Q), \vee, ^-, I \rangle$ is a complemented semilattice. Now, let $H(Q) = \{[g_q] \mid q \in Q\}$, where $g_q = (\{q\}, \emptyset)$.

Theorem 2 *$\langle E(Q), \vee, ^-, I\rangle$ is freely generated by $H(Q)$ in the class of complemented semilattices.*

Proof: Let $\langle S, \vee, ^-, 1\rangle$ be a complemented semilattice. Let $\mu : H(Q) \to S$ be any mapping. Extend μ to $E(Q)$ as follows. If $[(A, A')]$ is a singleton, let

$$\mu([(A, A')]) = \bigvee_{q \in A} \mu([g_q]) \vee \overline{\bigvee_{q \in A'} \mu([g_q])}.$$

Also let $\mu(I) = 1$. To verify that μ is a homomorphism, first consider the case where $[(A, A')]$ is a singleton. Then the argument in the proof on Proposition 1 applies, and

$$\mu(\overline{[(A, A')]}) = \overline{\mu([(A, A')])}.$$

Also

$$\mu(\overline{I}) = \mu(I) = 1 = \overline{1} = \overline{\mu(I)},$$

as required.

Note that

$$\mu([(A, A') \vee (B, B')]) = \mu([(A \cup B, A' \cup B')]).$$

If $[(A \cup B, A' \cup B')]$ is a singleton, then the argument of Proposition 1 applies and

$$\mu([(A, A') \vee (B, B')]) = \mu([(A, A')]) \vee \mu([(B, B')]).$$

Next, if one of the arguments is I, we have

$$\mu(I \vee [(A, A')]) = \mu(I) = 1 = 1 \vee \mu([(A, A')]) = \mu(I) \vee \mu([(A, A')]).$$

Finally, we have the case where $[(A, A')]$ and $[(B, B')]$ are both singletons, but $[(A \cup B, A' \cup B')] = I$. Then

$$\mu([(A, A') \vee (B, B')]) = \mu(I) = 1,$$

and

$$\mu([(A, A')] \vee \mu[(B, B')]) = \bigvee_{q \in A} \mu([g_q]) \vee \overline{\bigvee_{q \in A'} \mu([g_q])} \vee \bigvee_{q \in B} \mu([g_q]) \vee \overline{\bigvee_{q \in B'} \mu([g_q])}.$$

We know that $(A \cup B) \cap (A' \cup B') \neq \emptyset$; so suppose that $q \in (A \cup B) \cap (A' \cup B')$. Then the expression on the right must contain $\mu([g_q]) \vee \overline{\mu([g_q])}$, which is 1, since S is complemented. Hence

$$\mu([(A, A')]) \vee \mu([(B, B')]) = 1,$$

as required. □

Example 7 *The free complemented semilattice on zero generators is the semilattice* $\mathbf{S}_1$ *of Fig. 1. For one free generator, we have* $\mathbf{S}_{3.3}$ *of Fig. 1, and for two free generators, we have the semilattice of Fig. 6. In general, the free complemented semilattice on* n *free generators has* 3^n *elements. This follows because each generator can be chosen for the left component, or the right component, or not at all. This gives us* 3^n *elements, one of which is empty and not permitted. This empty element is therefore discarded, but* I *is added in its place.* □

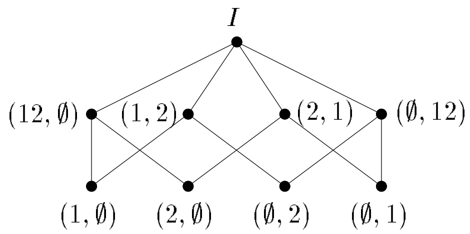

Figure 6: Free complemented semilattice E_2.

A finite involuted semilattice is *piecewise distributive*[5] if for each element $a \in S$, the lattice $a \vee S$ is distributive. The involuted semilattice of Fig. 3 is not piecewise distributive because $(1, \emptyset) \vee S$ contains M_5. A complemented semilattice that is not piecewise distributive because it contains N_5 is shown in Fig. 7.

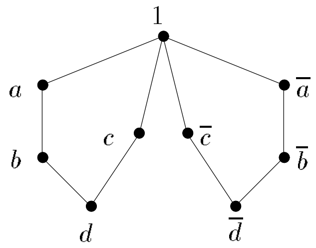

Figure 7: A semilattice that is not piecewise distributive.

[5]This notion differs from that of Grätzer's distributive semilattice.

Table 1: Equations of ternary algebra

T1 $a + a = a$	T1′ $a * a = a$
T2 $a + b = b + a$	T2′ $a * b = b * a$
T3 $a + (b + c) = (a + b) + c$	T3′ $a * (b * c) = (a * b) * c$
T4 $a + (a * b) = a$	T4′ $a * (a + b) = a$
T5 $a + 0 = a$	T5′ $a * 1 = a$
T6 $a + 1 = 1$	T6′ $a * 0 = 0$
T7 $\overline{(\overline{a})} = a$	
T8 $a + (b * c) = (a + b) * (a + c)$	T8′ $a * (b + c) = (a * b) + (a * c)$
T9 $\overline{(a + b)} = \overline{a} * \overline{b}$	T9′ $\overline{(a * b)} = \overline{a} + \overline{b}$
T10 $(a + \overline{a}) + \phi = a + \overline{a}$	T10′ $(a * \overline{a}) * \phi = a * \overline{a}$
T11 $\overline{\phi} = \phi$	

6 Ternary algebras

For a short history of ternary algebras see Brzozowski, Lou, and Negulescu [4]. Here we follow Brzozowski and Seger [5] in defining a ternary algebra as a de Morgan algebra with an additional constant ϕ satisfying $\phi = \overline{\phi}$ and $(a + \overline{a}) + \phi = a + \overline{a}$. More recently, free ternary algebras were studied by Balbes [1].

A *ternary algebra* is an algebra $\langle T, +, *, ^{-}, 0, \phi, 1\rangle$, where T is a set, $+$ and $*$ are binary operations (which we call *addition* and *multiplication*) on T, $^{-}$ is a unary operation on T, called *quasi-complementation*, 0, ϕ and 1 are constants in T, and the equations of Table 1 are satisfied for all a, b and c in T.

We define the partial order in a ternary algebra, as we do in any lattice:[6]

$$a \leq b \text{ iff } a + b = b.$$

This is equivalent to

$$a \leq b \text{ iff } a * b = a.$$

It was shown in [4] that

$$a \leq b \text{ iff } \overline{a} \geq \overline{b},$$

[6]Previously, we used $\preceq$ as the partial order of an upper semilattice. We reserve that symbol for another upper semilattice that will be associated with a ternary algebra.

and, in particular,

$$a \leq \phi \text{ iff } \overline{a} \geq \phi.$$

It was noted in [4] that every finite ternary algebra contains an odd number of elements, and that for each odd integer $n \geq 3$, there is at least one ternary algebra with n elements.

7 Subset-pair algebras

Let S be an arbitrary set, and $P(S)$, the set of all ordered pairs (A, A') of subsets of S such that $A \cup A' = S$. Define 0, ϕ and 1 as follows:

$$0 = (S, \emptyset),\ \phi = (S, S),\ 1 = (\emptyset, S).$$

Furthermore, define the following operations on $P(S)$:

$$\begin{aligned}(A, A') + (B, B') &= (A \cap B, A' \cup B'),\\ (A, A') * (B, B') &= (A \cup B, A' \cap B'),\\ \overline{(A, A')} &= (A', A).\end{aligned}$$

Let R be any subset of $P(S)$. Then $\langle R, +, *, ^-, 0, \phi, 1\rangle$ is a *subset-pair algebra* if R is closed under $+$, $*$ and $^-$, and contains 0, ϕ, and 1.

It is easy to verify that every subset-pair algebra is a ternary algebra, *i.e.,* satisfies the equations of Table 1. The converse result, that every ternary algebra is isomorphic to a subset-pair algebra has been proved by Brzozowski, Lou, and Negulescu [4] for the finite case, and by Ésik [6] for the infinite case. Thus we have

Theorem 3 *Every subset-pair algebra is a ternary algebra, and every ternary algebra is isomorphic to a subset-pair algebra.*

From now on we use this result freely, and usually assume that the ternary algebra we are studying has already been represented in the subset-pair notation. Thus, if a and b are elements of a ternary algebra T, we use (A, A') and (B, B') to denote the corresponding elements of the subset-pair algebra isomorphic to T, or simply write $a = (A, A')$ and $b = (B, B')$.

We have the following representation of the partial order $\leq$ in the subset-pair algebra:

$$(A, A') \leq (B, B') \text{ iff } A \supseteq B \text{ and } A' \subseteq B'.$$

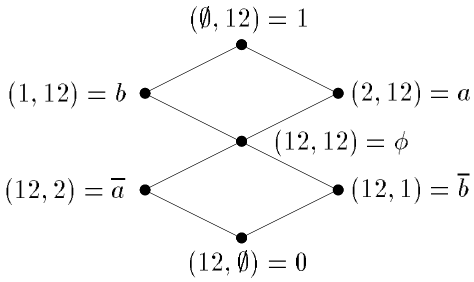

Figure 8: A subset-pair algebra $\mathbf{T}_7$.

Example 8 *Figure 8 shows a Hasse diagram of a subset-pair algebra with seven elements. We label the elements with their subset-pair representatives, and also with the symbols* $0, \overline{a}, \overline{b}, \phi, b, a, 1$, *as shown in the figure. The addition table for these elements is constructed by using the join, and the multiplication table, using the meet, as shown in Table 2. The unary operation* $^{-}$ *is clear from the notation, i.e., the complementary pairs are* $(a, \overline{a})$, $(b, \overline{b})$, $(0, 1)$, *and* (ϕ, ϕ). *We return to this example later.* □

Table 2: $+$ and $*$ operations for $\mathbf{T}_7$

$+$	0	$\overline{a}$	$\overline{b}$	ϕ	b	a	1
0	0	$\overline{a}$	$\overline{b}$	ϕ	b	a	1
$\overline{a}$	$\overline{a}$	$\overline{a}$	ϕ	ϕ	b	a	1
$\overline{b}$	$\overline{b}$	ϕ	$\overline{b}$	ϕ	b	a	1
ϕ	ϕ	ϕ	ϕ	ϕ	b	a	1
b	b	b	b	b	b	1	1
a	a	a	a	a	1	a	1
1	1	1	1	1	1	1	1

$*$	0	$\overline{a}$	$\overline{b}$	ϕ	b	a	1
0	0	0	0	0	0	0	0
$\overline{a}$	0	$\overline{a}$	0	$\overline{a}$	$\overline{a}$	$\overline{a}$	$\overline{a}$
$\overline{b}$	0	0	$\overline{b}$	$\overline{b}$	$\overline{b}$	$\overline{b}$	$\overline{b}$
ϕ	0	$\overline{a}$	$\overline{b}$	ϕ	ϕ	ϕ	ϕ
b	0	$\overline{a}$	$\overline{b}$	ϕ	b	ϕ	b
a	0	$\overline{a}$	$\overline{b}$	ϕ	ϕ	a	a
1	0	$\overline{a}$	$\overline{b}$	ϕ	b	a	1

8 Uncertainty partial order

Figure 9 shows the lattice order of the three-element ternary algebra, and also its *uncertainty* partial order [5], where ϕ represents the highest value (unknown or uncertain), and 0 and 1 are the known or certain values. It was noted in [5] that the least upper bound of $\{a, b\}$ in this partial order can be expressed as $a * b + (a + b) * \phi$. We now apply this operation to arbitrary ternary algebras.

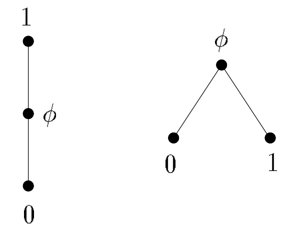

Figure 9: Ternary partial orders.

We use the convention that multiplication takes precedence over addition. In any ternary algebra $(T, +, *, ^-, 0, \phi, 1)$ define

$$a \vee b = a * b + (a + b) * \phi,$$

and, as before,

$$a \preceq b \text{ iff } a \vee b = b.$$

Proposition 5 *The algebra $(T, \vee, ^-, \phi)$ is a complemented semilattice with greatest element ϕ.*

Proof: It is easily verified that $(T, \vee, ^-, \phi)$ satisfies equations S1–S7. $\square$

It is also easy to see that we have the following representation in terms of the subset-pairs:

$$(A, A') \vee (B, B') = (A \cup B, A' \cup B'),$$

$$(A, A') \preceq (B, B') \text{ iff } A \subseteq B \text{ and } A' \subseteq B'.$$

Proposition 6 *Let a and b be elements of a ternary algebra T. Then*

1. *If $a \leq \phi$ then $a \vee b = a + b * \phi$.*

2. *If $a \geq \phi$ then $a \vee b = a * b + \phi$.*

3. *If $a, b \leq \phi$ then $a \vee b = a + b$.*

4. *If $a, b \geq \phi$ then $a \vee b = a * b$.*

5. *If $a \leq \phi \leq b$ then $a \vee b = \phi$.*

Proof: First, suppose $a \leq \phi$. Then $a \vee b = a*b+(a+b)*\phi = ab+a*\phi+b*\phi = a*b+a+b*\phi = a+b*\phi$.

Second, if $a \geq \phi$, then $a \vee b = a*b+a*\phi+b*\phi = a*b+\phi+b*\phi = a*b+\phi$.

Third, if $a, b \leq \phi$, then $a \vee b = a*b+a*\phi+b*\phi = a*b+a+b = a+b$.

Fourth, if $a \geq \phi$, then $a \vee b = a*b+\phi$, as above. If also $b \geq \phi$, then

$$a \vee b = a*b+\phi = \overline{\overline{a}+\overline{b}}+\overline{\overline{\phi}} = \overline{(\overline{a}+\overline{b})*\phi} = \overline{\overline{a}*\phi+\overline{b}*\phi}$$

$$= \overline{\overline{a}*\overline{\phi}+\overline{b}*\overline{\phi}} = \overline{\overline{a+\phi}+\overline{b+\phi}} = \overline{\overline{a}+\overline{b}} = a*b.$$

Fifth, if $a \leq \phi \leq b$, then $a \vee b = a*b+a*\phi+b*\phi = a+a+\phi = a+\phi = a*\phi+\phi = \phi$. $\square$

Example 9 *Consider ternary algebra $\mathbf{T_7}$ defined in Fig. 8. It is clear from the figure that $0, \overline{a}, \overline{b} < \phi$, and $1, a, b > \phi$. Let $T_{\leq\phi} = \{e \mid e \leq \phi\} = \{0, \overline{a}, \overline{b}, \phi\}$, and $T_{\geq\phi} = \{e \mid e \geq \phi\} = \{1, a, b, \phi\}$. Using Proposition 6, we immediately obtain Table 3 of the $\vee$ operation for $\mathbf{T_7}$, from Table 2. We have $a \vee b = a+b$ if $a, b \in T_{\leq\phi}$, $a \vee b = a*b$ if $a, b \in T_{\geq\phi}$, and $a \vee b = \phi$, otherwise.*

The partial order $\preceq$ is shown in Fig 4. It is the second semilattice with seven elements, where $c = 1$ and $\overline{c} = 0$. This semilattice is also isomorphic to the one of Fig. 5. $\square$

Example 10 *Consider the Hasse diagram in Fig. 10 (left part) of the ternary algebra $\mathbf{T}_{11}$. This is the free ternary algebra on one generator [1]. Let $T_{\leq\phi} = \{0, \overline{a}, \overline{b}, \overline{c}, \phi\}$, and $T_{\geq\phi} = \{1, a, b, c, \phi\}$. By Proposition 6, $x \vee y = x+y$ whenever $x \in T_{\leq\phi}$ and $y \in T_{\leq\phi}$, $x \vee y = x*y$ whenever $x \in T_{\geq\phi}$ and $y \in T_{\geq\phi}$, and $x \vee y = \phi$ whenever $x \in T_{\leq\phi}$ and $y \in T_{\geq\phi}$, or $x \in T_{\geq\phi}$ and $y \in T_{\leq\phi}$, or $x = \phi$ or $y = \phi$. Thus, we only have to calculate the entries involving d and $\overline{d}$ to obtain the table of the operation $\vee$, as shown in Table 4.*

The partial order $\preceq$ for $\mathbf{T}_{11}$ is shown in Fig. 10 (right part). $\square$

Table 3: $\vee$ operation for $\mathbf{T_7}$

$\vee$	0	$\overline{a}$	$\overline{b}$	ϕ	b	a	1
0	0	$\overline{a}$	$\overline{b}$	ϕ	ϕ	ϕ	ϕ
$\overline{a}$	$\overline{a}$	$\overline{a}$	ϕ	ϕ	ϕ	ϕ	ϕ
$\overline{b}$	$\overline{b}$	ϕ	$\overline{b}$	ϕ	ϕ	ϕ	ϕ
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
b	ϕ	ϕ	ϕ	ϕ	b	ϕ	b
a	ϕ	ϕ	ϕ	ϕ	ϕ	a	a
1	ϕ	ϕ	ϕ	ϕ	b	a	1

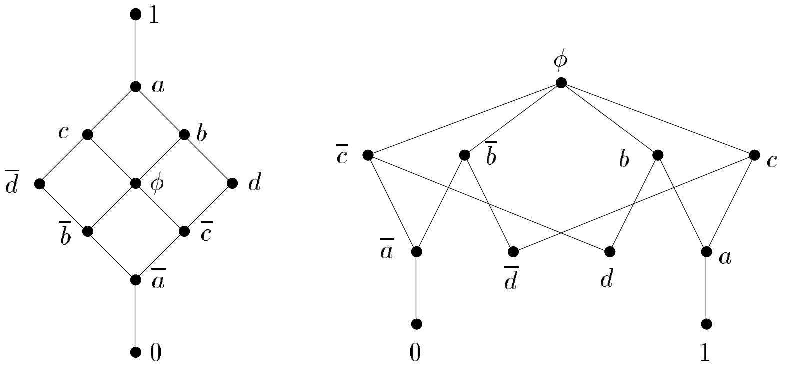

Figure 10: Hasse diagram for $\mathbf{T}_{11}$ and its $\preceq$ order.

Some additional properties of the operation $\vee$ are shown below; they are easily verified.

V1 $a * (b \vee c) = (a * b) \vee (a * c)$
V2 $a + (b \vee c) = (a + b) \vee (a + c)$
V3 $a * (a \vee b) = a + (a \vee b) = a \vee (a + b) = a + (b * \phi)$
V4 $a \vee (a * b) = a * (b + \phi)$

Acknowledgment

I am greatly indebted to Zoltán Ésik of Szeged University for significantly improving the terminology, the presentation of the results, and the proofs in this paper.

Table 4: Operation $\vee$ for T_{11}

$\vee$	0	$\overline{a}$	$\overline{b}$	$\overline{c}$	$\overline{d}$	ϕ	d	c	b	a	1
0	0	$\overline{a}$	$\overline{b}$	$\overline{c}$	$\overline{b}$	ϕ	$\overline{c}$	ϕ	ϕ	ϕ	ϕ
$\overline{a}$	$\overline{a}$	$\overline{a}$	$\overline{b}$	$\overline{c}$	$\overline{b}$	ϕ	$\overline{c}$	ϕ	ϕ	ϕ	ϕ
$\overline{b}$	$\overline{b}$	$\overline{b}$	$\overline{b}$	ϕ	$\overline{b}$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\overline{c}$	$\overline{c}$	$\overline{c}$	ϕ	$\overline{c}$	ϕ	ϕ	$\overline{c}$	ϕ	ϕ	ϕ	ϕ
$\overline{d}$	$\overline{b}$	$\overline{b}$	$\overline{b}$	ϕ	$\overline{d}$	ϕ	ϕ	c	ϕ	c	c
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
d	$\overline{c}$	$\overline{c}$	ϕ	$\overline{c}$	ϕ	ϕ	d	ϕ	b	b	b
c	ϕ	ϕ	ϕ	ϕ	c	ϕ	ϕ	c	ϕ	c	c
b	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	b	ϕ	b	b	b
a	ϕ	ϕ	ϕ	ϕ	c	ϕ	b	c	b	a	a
1	ϕ	ϕ	ϕ	ϕ	c	ϕ	b	c	b	a	1

References

[1] R. Balbes, "Free Ternary Algebras," *Int. J. Algebra and Computation*, vol. 10, no. 6, pp. 739–749, 2000.

[2] D. A. Bredikhin, "Ordered involuted semigroups of quasicomplete binary relations," pp. 3–11 in *Theory of semigroups and its applications, No. 4* (Russian) Saratov. Gos. Univ., Saratov, 1978. (Mathematical Reviews Number 80k:30048)

[3] D. A. Bredikhin, "On semilattice ordered involuted monoids of binary relations," pp. 11–15 in *Semigroups, automata and languages (Porto, 1994)* World Sci. Publishing, 1996.

[4] J. A. Brzozowski, J. J. Lou, and R. Negulescu, "A Characterization of Finite Ternary Algebras," *Int. J. Algebra and Computation*, vol. 7, no. 6, pp. 713–721, 1997.

[5] J. A. Brzozowski and C-J. H. Seger, *Asynchronous Circuits*, Springer-Verlag, 1995.

[6] Z. Ésik, "A Cayley Theorem for Ternary Algebras," *Int. J. Algebra and Computation*, vol. 8, no. 3, pp. 311–316, 1998.

[7] G. Grätzer, *Lattice Theory*, W. H. Freeman and Co., 1971.

[8] A. Horn, and N. Kimura, "The Category of Semilattices," *Algebra Universalis*, vol. 1, pp. 26–38, 1971.