De Morgan Demilattices*

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October 18, 1999

Abstract. We study algebras of the form $(S, \sqcup, \wedge)$, where S is a set and $\sqcup$ and $\wedge$ are binary operations on S such that $(S, \sqcup)$ and $(S, \wedge)$ are semilattices. These algebras can be viewed as double semilattices, and are generalizations of lattices, since they satisfy all the lattice laws except the absorption laws. For these reasons, we coin the word 'demilattice' for such structures. In particular, we study de Morgan demilattices, which are algebras of the form $(S, \sqcup, \wedge, ^-, 1, 0)$, where $(S, \sqcup, \wedge)$ is a demilattice, 1 and 0 are the unit and zero elements of S, and $^-$ is a unary operation, called quasi-complementation, that satisfies the involution law and de Morgan's laws. De Morgan demilattices are generalizations of de Morgan algebras. They have applications in multi-valued simulations of digital circuits.

1 A Quinary Algebra for Logic Circuits

In 1972 Lewis [5, 6] introduced a quinary, that is, five-valued, algebra for simulating the behavior of logic circuits. The five values used were the constant values 0 and 1, the values 0/1 and 1/0 indicating a signal changing from 0 to 1, and from 1 to 0, respectively, and X representing an indeterminate or unknown value. For convenience, we replace 0/1 by U (for *up*) and 1/0 by D (for *down*). In this model, the behaviors of the INVERTER, OR gate and AND gate are described by the operators $^-$, $\sqcup$, and $\wedge$, respectively, where 0 is the complement of 1, U is the complement of D, X is its own complement, and $\sqcup$, and $\wedge$ are defined in the tables below.

*This research was supported by the Natural Sciences and Engineering Research Council of Canada under grant No. OGP0000871.

Table 1: Operation of an OR gate.

$\sqcup$	1	X	D	U	0
1	1	1	1	1	1
X	1	X	X	X	X
D	1	X	D	X	D
U	1	X	X	U	U
0	1	X	D	U	0

Table 2: Operation of an AND gate.

$\wedge$	0	X	U	D	1
0	0	0	0	0	0
X	0	X	X	X	X
U	0	X	U	X	U
D	0	X	X	D	D
1	0	X	U	D	1

It is easily verified that the three operations obey the following laws:

L1	$x \sqcup x = x$	L1′	$x \wedge x = x$
L2	$x \sqcup y = y \sqcup x$	L2′	$x \wedge y = y \wedge x$
L3	$x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$	L3′	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$
L4	$x \sqcup 1 = 1$	L4′	$x \wedge 0 = 0$
L5	$x \sqcup 0 = x$	L5′	$x \wedge 1 = x$
L6	$\overline{\overline{x}} = x$		
L7	$\overline{x \sqcup y} = \overline{x} \wedge \overline{y}$	L7′	$\overline{x \wedge y} = \overline{x} \sqcup \overline{y}$

The absorption laws do not hold because $U \wedge (U \sqcup D) = U \wedge X = X \neq U$, and the dual law is violated by the dual argument. Also, the distributive laws do not hold since

$$U \wedge (D \sqcup 1) = U \wedge 1 = U \neq X = X \sqcup U = (U \wedge D) \sqcup (U \wedge 1),$$

and the dual argument applies to the dual law. The operation $^-$ is not a complement operation, since $U \wedge \overline{U} = U \wedge D = X \neq 0$, and, similarly, $U \sqcup \overline{U} \neq 1$.

In this paper we generalize this example, and study the properties of algebras of the form $(S, \sqcup, \wedge, ^-, 1, 0)$, in which S is a set, $\sqcup$ and $\wedge$ are binary operations on S, $^-$ is a unary operation on S, 1 and 0 are elements of S, and laws L1–L7, and their duals are satisfied.

2 Semilattices

We now briefly review basic properties of semilattices, and establish the notation for the remainder of the paper.

A partially ordered set $(S, \sqsupseteq)$ is 1-*bounded* if it has a *unit*, that is, an element 1 such that $1 \sqsupseteq x$ for all $x \in S$. It is 0-*bounded* if it has a *zero*, that is, an element 0 such that $x \sqsupseteq 0$ for all $x \in S$. A partial order is *bounded* if it is both 1-bounded and 0-bounded.

A partially ordered set $(S, \sqsupseteq)$ is an *upper semilattice* if every pair of elements has a least upper bound $lub_{\sqsupseteq}$. In an upper semilattice, the binary operation $\sqcup$ defined by

$$x \sqcup y = z \text{ iff } z = lub_{\sqsupseteq}\{x, y\}$$

satisfies L1–L3. Equivalently, any set S with a binary operation $\sqcup$ satisfying L1–L3, is an upper semilattice, since we can define the partial order $\sqsupseteq$ by

$$x \sqsupseteq y \text{ iff } x \sqcup y = x,$$

and then $x \sqcup y$ is the least upper bound of $\{x, y\}$.

An upper semilattice that is 1-bounded (respectively, 0-bounded) satisfies L4 (respectively, L5).

An upper semilattice $(S, \sqcup)$ is *complete* [1] if every subset of S has a least upper bound $lub_{\sqsupseteq}$. Note that in a complete upper semilattice $lub_{\sqsupseteq}\emptyset = 0$ and $lub_{\sqsupseteq}S = 1$. Thus a complete upper semilattice is always bounded. Also, every complete upper semilattice is a complete lattice [1]. For suppose that $(S, \sqcup)$ is a complete upper semilattice, and consider any $T \subseteq S$. Let L be the set of all lower bounds of T. Since S is complete, the element $z = lub_{\sqsupseteq}L$ exists. Now $z = glb_{\sqsupseteq}T$. If we now define

$$x \sqcap y = z \text{ iff } z = glb_{\sqsupseteq}\{x, y\},$$

we have the lattice $(S, \sqcup, \sqcap, 1, 0)$, which we call the lattice *induced* by the complete semilattice $(S, \sqcup, 1, 0)$.

Note that every finite bounded semilattice $(S, \sqsupseteq, 1, 0)$ is a lattice. For suppose that a and b are any two elements of S. Then a and b have at least one lower bound with respect to $\sqsupseteq$, namely 0. Thus the set L of all lower bounds of a and b is not empty, and has a least upper bound with respect to $\sqsupseteq$. This least upper bound is then the greatest lower bound of a and b.

Dually, a partially ordered set $(S, \leq)$ is a *lower semilattice* if every pair of elements has a greatest lower bound $glb_{\leq}$. In a lower semilattice, the binary operation $\wedge$ defined by

$$x \wedge y = z \text{ iff } z = glb_{\leq}\{x, y\}$$

satisfies L1′–L3′. Equivalently, any set S with a binary operation $\wedge$ satisfying L1′–L3′, is a lower semilattice, since we can define the partial order $\leq$ by

$$x \leq y \text{ iff } x \wedge y = x,$$

and then $x \wedge y$ is the greatest lower bound of $\{x, y\}$.

A lower semilattice that is 0-bounded (respectively, 1-bounded) satisfies L4′ (respectively, L5′).

A lower semilattice $(S, \leq)$ is *complete* if every subset of S has a greatest lower bound $glb_{\leq}$. Note that in a complete lower semilattice $glb_{\leq}\emptyset = 1$ and $glb_{\leq}S = 0$. Thus a complete lower semilattice is always bounded. Every complete lower semilattice is a complete lattice. For suppose that $(S, \wedge)$ is a lower semilattice, and consider any $T \subseteq S$. Let U be the set of all upper bounds of T. Since S is complete, the element $z = glb_{\leq}U$ exists. Now $z = lub_{\leq}T$. If we define

$$x \vee y = z \text{ iff } z = lub_{\leq}\{x, y\},$$

we have the lattice $(S, \wedge, \vee, 0, 1)$, which we call the lattice *induced* by the complete lower semilattice $(S, \wedge, 0, 1)$.

3 Demilattices

A *demilattice* $(S, \sqcup, \wedge)$, consists of a set S with two binary operations $\sqcup$ and $\wedge$ on S, such that $(S, \sqcup)$ and $(S, \wedge)$ are semilattices. In other words, $(S, \sqcup, \wedge)$ satisfies laws L1–L3, L1′–L3′. Two partial orders are associated with S, as follows. For all $x, y \in S$,

$$x \sqsupseteq y \text{ iff } x \sqcup y = x, \qquad x \leq y \text{ iff } x \wedge y = x.$$

As usual, we use $\sqsubseteq$ and $\geq$ for the converse relations. If two elements a and b of S are not related by $\sqsupseteq$ (respectively $\leq$) we call them $\sqsupseteq$-*incomparable* (respectively $\leq$-*incomparable*). Two elements that are both $\sqsupseteq$-incomparable and $\leq$-incomparable are called *incomparable*.

A demilattice $(S, \sqcup, \wedge)$ is *quasi-bounded* if $(S, \sqcup)$ is 1-bounded and $(S, \wedge)$ is 0-bounded. Equivalently, $(S, \sqcup, \wedge)$ is quasi-bounded if it has elements $1_{\sqcup}$ and $0_{\wedge}$ satisfying L4 and L4′. Every finite demilattice is quasi-bounded.

In a demilattice, the two component semilattices are related only by the fact that they share the same underlying set. If the absorption laws

$$\text{L8} \quad x \sqcup (x \wedge y) = x \qquad \text{L8}' \quad x \wedge (x \sqcup y) = x$$

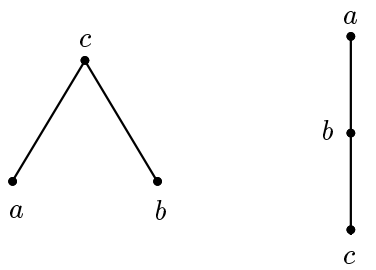
are satisfied, then the demilattice is a lattice, and $x \wedge y$ is the $glb_{\sqsupseteq}\{x, y\}$, that is, $x \wedge y = x \sqcap y$. Similarly, $x \vee y = x \sqcup y$. To verify this, note that $x = x \sqcup (x \wedge y)$ by the absorption law. Thus, $x \sqsupseteq x \wedge y$, and, similarly, $y \sqsupseteq x \wedge y$. Consequently, $x \wedge y$ is a lower bound of x and y with respect to $\sqsupseteq$. If z is any other lower bound of x and y with respect to $\sqsupseteq$, then $x \sqcup z = x$ and $y \sqcup z = y$. Then $z = z \wedge (x \sqcup z) = z \wedge x$, $z = z \wedge y$, and $z = z \wedge y = (z \wedge x) \wedge y = z \wedge (x \wedge y)$. Therefore, $(x \wedge y) \sqcup z = (x \wedge y) \sqcup (z \wedge (x \wedge y)) = x \wedge y$ by absorption. Hence $x \wedge y \sqsupseteq z$, and $x \wedge y$ is indeed the greatest lower bound. Dually, $x \sqcup y$ is the least upper bound with respect to $\leq$.

Also, if the absorption laws are satisfied and the two semilattices are bounded, then $1_{\sqcup} = 1_{\wedge} \sqcup 1_{\sqcup} = 1_{\wedge} \sqcup (1_{\wedge} \wedge 1_{\sqcup}) = 1_{\wedge}$, and, dually, $0_{\sqcup} = 0_{\wedge}$.

A demilattice $(S, \sqcup, \wedge)$ is a *quasi-lattice* if $(S, \sqcup)$ $(S, \wedge)$ are lattices. The lattices are then denoted by $(S, \sqcup, \sqcap)$ and $(S, \wedge, \vee)$.

A demilattice $(S, \sqcup, \wedge)$ is *complete* if $(S, \sqcup)$ and $(S, \wedge)$ are complete. Every finite lattice is complete. A complete demilattice $(S, \sqcup, \wedge, 1_\sqcup, 0_\wedge)$ has induced lattices $(S, \sqcup, \sqcap, 1_\sqcup, 0_\sqcup)$ and $(S, \wedge, \vee, 0_\wedge, 1_\wedge)$.

Example 1 *Figure 1 shows the two partial orders $\sqsupseteq$ and $\leq$ of a three-element demilattice $(S, \sqcup, \wedge)$, with $S = \{a, b, c\}$. The semilattice $(S, \sqcup)$ is not a lattice; it is 1-bounded, with $1_\sqcup = c$; it is not complete, since a and b have no lower bound. The semilattice $(S, \wedge)$ is a lattice; it is bounded with $0_\wedge = c$, and $1_\wedge = a$. Note that $0_\wedge = 1_\sqcup = c$.* □



(a) (b)

Figure 1: Partial orders for a demilattice: (a) $\sqsupseteq$; (b) $\leq$.

Example 2 *Figure 2 shows the partial orders $\sqsupseteq$ and $\leq$ associated with the operations $\sqcup$ and $\wedge$ of Tables 2 and 3. Here, $1_\sqcup = 1_\wedge = 1$ and $0_\sqcup = 0_\wedge = 0$, but the absorption laws do not hold. Here we have a quasi-lattice; in fact, both semilattices are distributive lattices. Note that $U \sqcap D = 0$, but $U \wedge D = X$.* □

Proposition 1 *Let $(S, \sqcup, \wedge)$ be a demilattice and let n be the cardinality of S. Let $N_d(n)$ and $N_q(n)$ be the numbers of non-isomorphic demilattices and quasi-lattices with n elements, respectively. Then $N_d(1) = N_q(1) = 1$, $N_d(2) = N_q(2) = 2$, $N_d(3) = 18$ and $N_q(3) = 8$.*

Proof: First consider the number of distinct demilattices. If S has one element, we have the trivial one-element lattice.

If S has two elements, we have two cases. If $1_\sqcup \neq 0_\wedge$, then $S = \{1_\sqcup, 0_\wedge\}$. If we simplify the notation to $S = \{1, 0\}$, for the $\sqcup$ operation we must have $1 \sqcup 1 = 1$ and $0 \sqcup 0 = 0$ by L1, $0 \sqcup 1 = 1$ by L4, and $1 \sqcup 0 = 0 \sqcup 1 = 1$ by L2. This is Boolean

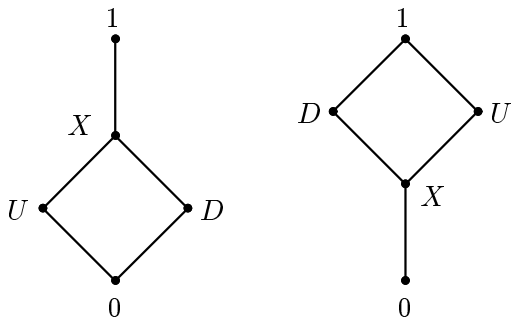


Figure 2: Partial orders for quinary algebra: (a) $\sqsupseteq$; (b) $\leq$.

addition, which is associative. Similarly, $\wedge$ must be Boolean multiplication. Here S is the two-element lattice.

If $1_\sqcup = 0_\wedge$, use a to represent $1_\sqcup = 0_\wedge$, and let b be the second element. We must have $b \sqcup b = b$, $a \sqcup a = a \sqcup b = b \sqcup a = a$, and $b \wedge b = b$, $a \wedge a = a \wedge b = b \wedge a = a$. Thus the two operations are identical. Note that $(S, \sqcup, \wedge)$ is a quasi-lattice. Here $a \sqsupseteq b$ and $a \leq b$.

If S has three elements, again we have two cases. If $1_\sqcup = 1 \neq 0_\wedge = 0$, let a be the third element. We must have $1 \sqcup 1 = 1$, $a \sqcup a = a$, and $0 \sqcup 0 = 0$ by L1; $0 \sqcup 1 = a \sqcup 1 = 1$ by L4; $1 \sqcup 0 = 1 \sqcup a = 1$ by L4; and $0 \sqcup a = a \sqcup 0$ by L2. There are three choices for $0 \sqcup a = a \sqcup 0$, namely 1, a, and 0. One verifies that all the choices lead to an associative addition. These choices correspond to the three partial orders (1) $1 \sqsupseteq a, 0$ and a and 0 are $\sqsupseteq$-incomparable, 2) $1 \sqsupseteq a \sqsupseteq 0$, and (3) $1 \sqsupseteq 0 \sqsupseteq a$. Since dual arguments apply to the $\wedge$ operation, we can have (1) $0 \leq a, 1$ and a and 1 are $\leq$-incomparable, (2) $0 \leq a \leq 1$, and (3) $0 \leq 1 \leq a$. Hence, there are altogether nine possibilities.

In case $1_\sqcup = 0_\wedge$, let the other two elements be a and b. All the entries in the addition and multiplication tables are uniquely determined except the entries for $a \sqcup b = b \sqcup a$ and $a \wedge b = b \wedge a$. As in the case above, there are three choices for $a \sqcup b$ and three choices for $a \wedge b$, again resulting in nine cases.

All three demilattices with $n \leq 2$ are quasi-lattices. For $n = 3$ one verifies that only eight of the demilattices are quasi-lattices. □

4 Consistently Bounded Demilattices

A *consistently bounded* demilattice is a demilattice with 0 and 1 satisfying L4–L5, L4′–L5′. From now on we consider only consistently bounded demilattices with at least two elements.

Proposition 2 *Let $(S, \sqcup, \wedge, 1, 0)$ be a consistently bounded demilattice.*

(a) *For all $x, y \in S$, $x \sqcup y = 0$ implies $x = y = 0$, and $x \wedge y = 1$ implies $x = y = 1$.*

(b) *If S is consistently bounded and has at least two elements, then $0 \neq 1$.*

(c) *Let $N_{cb}(n)$ be the number of consistently bounded demilattices with n elements. Then $N_{cb}(2) = N_{cb}(3) = 1$, and $N_{cb}(4) = 4$.*

Proof: For (a), suppose x is any element different from 0, and that $x \sqcup y = 0$. Then $x = x \sqcup 0 = x \sqcup (x \sqcup y) = (x \sqcup x) \sqcup y = x \sqcup y = 0$, which is a contradiction. A dual argument proves the second claim. For (b), suppose $0 = 1$, and a is any other element; then $a = a \sqcup 0 = a \sqcup 1 = 1$, which is a contradiction.

For (c), if $n = 2$, only the quasi-bounded demilattice with $0 \neq 1$ is consistently bounded.

For $n = 3$, let the underlying set be $S = \{0, a, 1\}$. The $\sqcup$ and $\wedge$ tables are uniquely defined by the laws. This is the three-element lattice.

For $n = 4$, let the underlying set be $S = \{0, a, b, 1\}$. The $\sqcup$ and $\wedge$ tables are uniquely defined by the laws, except for the entries $a \sqcup b = b \sqcup a$ and $a \wedge b = b \wedge a$. In view of Proposition 2(a), there are three possibilities for each entry, giving a total of 9 cases to be considered.

If $0 \sqsubseteq a, b \sqsubseteq 1$ and $0 \leq a, b \leq 1$, we have the four-element lattice in which a and b are incomparable.

If $0 \sqsubseteq a \sqsubseteq b \sqsubseteq 1$ and $0 \leq a \leq b \leq 1$, we have the four-element lattice which is totally ordered. The case $0 \sqsubseteq b \sqsubseteq a \sqsubseteq 1$ and $0 \leq b \leq a \leq 1$ is isomorphic.

The case $0 \sqsubseteq a \sqsubseteq b \sqsubseteq 1$ and $0 \leq a, b \leq 1$ is isomorphic to $0 \sqsubseteq b \sqsubseteq a \sqsubseteq 1$ and $0 \leq a, b \leq 1$. The case $0 \sqsubseteq a, b \sqsubseteq 1$ and $0 \leq a \leq b \leq 1$ is isomorphic to $0 \sqsubseteq a, b \sqsubseteq 1$ and $0 \leq b \leq a \leq 1$. The last two cases can be obtained from the first two by interchanging the two operations.

The case $0 \sqsubseteq a \sqsubseteq b \sqsubseteq 1$ and $0 \leq b \leq a \leq 1$ is isomorphic to $0 \sqsubseteq b \sqsubseteq a \sqsubseteq 1$ and $0 \leq a \leq b \leq 1$. □

Note that all consistently bounded finite demilattices are necessarily quasi-lattices.

5 De Morgan Demilattices

A *de Morgan demilattice* is an algebra $(S, \sqcup, \wedge, ^-, 1, 0)$, where $(S, \sqcup, \wedge, 1, 0)$ is a consistently bounded demilattice and $^-$ is a unary operation on S, called *quasi-complement*, which satisfies the involution law L6 and de Morgan's laws L7, L7′.

An element a of a de Morgan demilattice is *self-complementary* if $\overline{a} = a$.

Proposition 3 *The following holds in any de Morgan demilattice $(S, \sqcup, \wedge, ^-, 1, 0)$:*

(a) $\overline{0} = 1$*;*

(b) $a \leq b$ *iff* $\overline{a} \sqsupseteq \overline{b}$*;*

(c) *if* $a \neq b$, $a \sqsubseteq b$, *and* $a \leq b$, *then either* $a \neq \overline{a}$ *or* $b \neq \overline{b}$;

(d) *if* $a \neq b$, $a \sqsupseteq b$, *and* $a \leq b$, *then* $a \neq \overline{b}$;

(e) *Let* $N_{dm}(n)$ *be the number of de Morgan demilattices with* n *elements. Then* $N_{dm}(2) = N_{dm}(3) = 1$, *and* $N_{dm}(4) = 3$.

Proof: For (a), suppose $\overline{0} = a$. Then $0 = \overline{a}$ and $1 = 1 \sqcup a = \overline{\overline{1 \sqcup a}} = \overline{\overline{1} \wedge \overline{a}} = \overline{\overline{1} \wedge 0} = \overline{0}$, showing that $a = 1$. For (b), suppose $a \leq b$; then $a \wedge b = a$, and $\overline{a} = \overline{a \wedge b} = \overline{a} \sqcup \overline{b}$, that is $\overline{a} \sqsupseteq \overline{b}$. For (c), suppose $a \neq b$, $a \sqsubseteq b$, $a \leq b$, $a = \overline{a}$, and $b = \overline{b}$. Then $a = a \wedge b = \overline{a} \wedge \overline{b} = \overline{a \sqcup b} = \overline{b} = b$, which is a contradiction. For (d), suppose $a \neq b$, $a \sqsupseteq b$, $a \leq b$, and $a = \overline{b}$. Then, by Proposition 3(b), $\overline{a} \sqsupseteq \overline{b}$, and so $b \sqsupseteq a$, which is a contradiction.

For (e), if $n = 2$, by Proposition 3(a), we have the two-element Boolean algebra.

If $n = 3$, let $S = \{0, a, 1\}$. Since there is only one consistently bounded demilattice, it only remains to determine the quasi-complements. Since $\overline{0} = 1$, we must have $\overline{a} = a$. Here we have the three-element ternary algebra [3, 4].

For $n = 4$, let $S = \{0, a, b, 1\}$. Suppose a and b are incomparable in both partial orders. By Proposition 2(a) we must have $a \sqcup b = 1$ and $a \wedge b = 0$. If $a = \overline{b}$, we have the four-element Boolean algebra.

If a and b are incomparable, and a and b are self-complementary, we have the four-element subdirectly irreducible de Morgan algebra [1].

Suppose $a \leq b$ and $a \sqsubseteq b$. By Proposition 3(c), either a or b is not self-complementary. Suppose a is that element; then b is the only possible quasi-complement of a. The same is true if b is not self-complementary. This de Morgan demilattice is isomorphic to the de Morgan algebra $(\{0, .25, .75, 1\}, \max, \min, \bar{}, 1, 0)$, where max and min are the maximum and minimum operations, and $\overline{x} = 1 - x$.

The case where $a \leq b$, but a and b are $\sqsupseteq$-incomparable cannot occur. For suppose $a = \overline{a}$ and $b = \overline{b}$. Since $a \leq b$, we have $\overline{a} \sqsupseteq \overline{b}$, by Proposition 3(b). But then $a \sqsupseteq b$, which is a contradiction. The only other possibility is $a = \overline{b}$. Since $a \leq b$, we have $\overline{a} \sqsupseteq \overline{b}$, that is, $b \sqsupseteq a$, which is again a contradiction.

The case $a \leq b$ and $a \sqsupseteq b$ cannot occur. By Proposition 3(d), $a \neq \overline{b}$. Hence $a = \overline{a}$ and $b = \overline{b}$ are the only possibilities. But then $b = \overline{a} = \overline{a \wedge b} = \overline{a} \sqcup \overline{b} = a \sqcup b = a$, which is a contradiction. □

The proposition above shows that, for $n \leq 4$, every de Morgan semilattice is a de Morgan algebra. The example of Section 1 shows that there is a five-element de Morgan semilattice which is not a de Morgan algebra. De Morgan demilattices with 2, 3, and 4 elements are shown in Figure 3. Self-complementary elements are shown in boldface type.

6 Quasi-Distributive De Morgan Demilattices

A de Morgan demilattice $(S, \sqcup, \wedge, ^-, 1, 0)$ is *quasi-distributive* if the induced lattices $(S, \sqcup, \sqcap, 1, 0)$ and $(S, \wedge, \vee, 0, 1)$ are distributive. We now derive a characterization

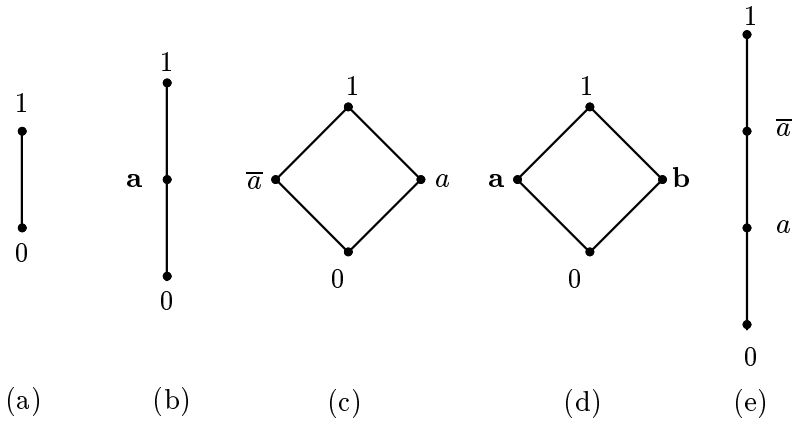


Figure 3: De Morgan demilattices with 2, 3, and 4 elements.

of de Morgan demilattices which is somewhat similar to the characterization of de Morgan algebras described in [2].

Suppose we are given a de Morgan demilattice $(S, \sqcup, \wedge, ^-, 1, 0)$. Since the lattice $(S, \sqcup, \sqcap, 1, 0)$ is distributive, by Birkhoff's theorem [1], it can be embedded in a ring of sets. Thus there exists a set I such that $(S, \sqcup, \sqcap, 1, 0)$ is isomorphic to $(X, \cap, \cup, \emptyset, I)$, under the correspondence $s \mapsto \sigma(s)$, where X is a set of subsets of I, and $\sigma(s)$ denotes the set assigned to s by the embedding. Here $\sqcup$ corresponds to intersection, $\sqcap$ to union, 1 to $\emptyset$, and 0 to I. In general, X is not closed under set complementation.

We can also assign to each element $s \in S$ the set $\overline{\sigma}(s) = \sigma(\overline{s})$. In a de Morgan demilattice, the assignment $x \mapsto \overline{x}$ satisfies $x \leq y$ iff $\overline{x} \sqsupseteq \overline{y}$. Now $x \wedge y = z$ iff $\overline{x \wedge y} = \overline{z}$ iff $\overline{x} \sqcup \overline{y} = \overline{z}$ iff $\sigma(\overline{x}) \cap \sigma(\overline{y}) = \sigma(\overline{z})$ iff $\overline{\sigma}(x) \cap \overline{\sigma}(y) = \overline{\sigma}(z)$. Thus, $(S, \wedge, \vee, 0, 1)$ is isomorphic to $(X, \cap, \cup, \emptyset, I)$ under the correspondence $s \mapsto \overline{\sigma}(s)$. Here $\wedge$ corresponds to intersection, $\vee$ to union, 0 to $\emptyset$, and 1 to I.

If we use the double embedding, where each element $s \in S$ corresponds to an ordered pair $(\sigma(s), \overline{\sigma}(s))$ of subsets of I, we have the following properties. For all $x, y, z \in S$,

$$x \sqcup y = z \text{ iff } \sigma(x) \cap \sigma(y) = \sigma(z),$$
$$x \wedge y = z \text{ iff } \overline{\sigma}(x) \cap \overline{\sigma}(y) = \overline{\sigma}(z),$$
$$\overline{x} = y \text{ iff } (\sigma(x), \overline{\sigma}(x)) = (\overline{\sigma}(y), \sigma(y)).$$

Example 3

For the five-element semilattice of Section 1, one assignment is $0 \mapsto (\{1,2,3\}, \emptyset)$, $X \mapsto (\{1\}, \{1\})$, $U \mapsto (\{1,2\}, \{1,3\})$, $D \mapsto (\{1,3\}, \{1,2\})$, $1 \mapsto (\emptyset, \{1,2,3\})$, as shown in Figure 4.

We can use this representation to simulate circuit behavior as follows. Suppose x is the input of an inverter and y is its output. If x is represented by $(\sigma(x), \overline{\sigma}(x))$,

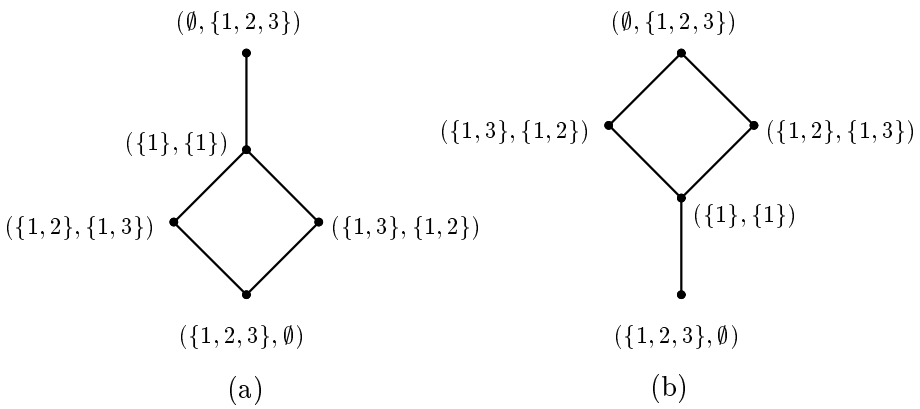


Figure 4: Set embedding for quinary algebra: (a) $\sqsupseteq$; (b) $\leq$.

then the output of the inverter is calculated by simply changing the order of the two components, that is y corresponds to $(\overline{\sigma}(x), \sigma(x))$. Next, suppose the inputs to an OR gate are $(\sigma(x), \overline{\sigma}(x))$ and $(\sigma(y), \overline{\sigma}(y))$. The first component of the output is then $C = \sigma(x) \cap \sigma(y)$. The second component can be calculated by table look-up. First find the element of S corresponding to C. There is exactly one such element, since the mapping σ is an isomorphism, and this element is $\sigma^{-1}(C)$. We can then apply the second mapping $\overline{\sigma}$. In summary,

$$(A, B) \sqcup (A', B') = (A \cap A', \overline{\sigma}(\sigma^{-1}(A \cap A'))),$$

$$(A, B) \wedge (A', B') = (\sigma(\overline{\sigma}^{-1}(B \cap B')), B \cap B'),$$

and

$$\overline{(A, B)} = (B, A).$$

The dual set assignment, where both $\sqcup$ and $\wedge$ are represented by union, is $0 \mapsto (\emptyset, \{1, 2, 3\})$, $X \mapsto (\{1, 2\}, \{1, 2\})$, $U \mapsto (\{1\}, \{2\})$, $D \mapsto (\{2\}, \{1\})$, $1 \mapsto (\{1, 2, 3\}, \emptyset)$.

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