Free De Morgan Bisemigroups and Bisemilattices

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Abstract

We give a geometric representation of free De Morgan bisemigroups, free commutative De Morgan bisemigroups and free De Morgan bisemilattices, using labeled graphs.

1 Introduction

J. A. Brzozowski and Z. Ésik have introduced in [3] algebra C capable of representing and counting hazards in asynchronous circuits. Algebra C has two binary operations $\oplus$ and $\otimes$, a unary operation $^-$, called quasi-complementation, and the constants 0 and 1 such that both $(C, \oplus, 0)$ and $(C, \otimes, 1)$ are commutative monoids, $^-$ is an involution satisfying De Morgan's law with respect to the operations $\oplus$ and $\otimes$, and such that $\overline{0} = 1$. In [3], such algebras were called commutative De Morgan bisemigroups, a generalization of De Morgan bisemilattices studied, in connection with circuits, in [2]. We conjecture that the variety of De Morgan bisemigroups is in fact generated by algebra C, so that an equation holds in C if and only if it is provable from the defining equations of De Morgan bisemigroups.

In this paper, building on the geometric description of free commutative bisemigroups and free bisemigroups [9, 8, 6], we provide a concrete geometric description of the free De Morgan bisemigroups, free commutative De Morgan bisemigroups, and free De Morgan bisemilattices, using labeled graphs and digraphs. In particular, we show that the free De Morgan bisemigroup on a set A may be represented as an algebra of isomorphism classes of $A \cup \overline{A}$-labeled sets, where $\overline{A}$ is a disjoint copy of A, equipped with two transitive digraph structures, in fact two N-free partial orders, such that any two elements of the set are related by exactly one of the two orders. The two binary operations are the series products with respect to the two orders, and the operation of quasi-complementation exchanges the two orders and complements the labels. The free commutative De Morgan bisemigroup on A has a similar description using $A \cup \overline{A}$-labeled graphs. Our study of algebras of labeled graphs, posets and biposets is also related to recent work on two-dimensional extensions of automata theory by Lodaya, Weil, Hashiguchi, Kuske and others, see [13, 14, 11, 12, 7].

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2 Preliminaries

Recall that *bisemigroup*[1] is an algebra $B = (B, \oplus, \otimes)$ equipped with binary associative operations $\oplus$ and $\otimes$. A *commutative bisemigroup* is a bisemigroup in which both operations are commutative. A *bisemilattice*[2] [2] is a commutative bisemigroup in which both operations are idempotent. In a bisemilattice, we will sometimes denote the operations by $\sqcup$ and $\wedge$. Morphisms of bisemigroups preserve the operations.

A *De Morgan bisemigroup* [2] is an algebra $D = (D, \oplus, \otimes, ^-, 0, 1)$ such that $(D, \oplus, \otimes)$ is a bisemigroup, and the *quasi-complementation* operation $^- : D \to D$ and the constants $0, 1$ satisfy

$$x \oplus 0 = 0 \oplus x = x \tag{1}$$
$$x \otimes 1 = 1 \otimes x = x \tag{2}$$
$$x \otimes 0 = 0 \otimes x = 0 \tag{3}$$
$$x \oplus 1 = 1 \oplus x = 1 \tag{4}$$

and

$$\overline{\overline{x}} = x \tag{5}$$
$$\overline{x \oplus y} = \overline{x} \otimes \overline{y} \tag{6}$$
$$\overline{x \otimes y} = \overline{x} \oplus \overline{y}. \tag{7}$$

It then follows that

$$\overline{0} = 1 \tag{8}$$
$$\overline{1} = 0 \tag{9}$$

A *commutative De Morgan bisemigroup* is a De Morgan bisemigroup which is a commutative bisemigroup, and a *De Morgan bisemilattice* is a De Morgan bisemigroup which is a bisemilattice. Morphisms of De Morgan bisemigroups, commutative De Morgan bisemigroups and De Morgan bisemilattices also preserve the constants and the quasi-complementation. Note that any De Morgan bisemigroup is determined by the $\oplus$ and $^-$ operations and the constant 0.

In any bisemilattice $B = (B, \sqcup, \wedge)$, the binary operations determine two partial orders $\sqsupseteq$ and $\leq$ defined by $x \sqsupseteq y$ if and only if $x \sqcup y = x$ and $x \leq y$ if and only if $x \wedge y = x$, for all $x, y \in B$. The operations are in turn determined by the partial orders in that $x \sqcup y$ is the l.u.b. of x and y with respect to the partial order $\sqsupseteq$, and $x \wedge y$ is the g.l.b. of x and y with respect to $\leq$. It is known that a bisemilattice is a *lattice* if and only if the partial orders $\sqsupseteq$ and $\leq$ coincide. When B is a De Morgan bisemilattice, 0 is least and 1 is greatest with respect to both partial orders. It is clear that any homomorphism of bisemilattices preserves $\sqsupseteq$ and $\leq$. Moreover, for x, y in a De Morgan bisemilattice,

$$x \sqsupseteq y \quad \Leftrightarrow \quad \overline{x} \leq \overline{y}. \tag{10}$$

This latter property is characteristic: If B is a bisemilattice equipped with a unary operation $^-$ and constants 0 and 1 satisfying (1) – (4), then B is a De Morgan bisemilattice if and only if (5) and (10) hold.

For all undefined notions of universal algebra, see any standard text such as [10, 4].

[1] Bisemigroups with a a common neutral element for the two associative operations were called *double monoids* in [9] and *bimonoids* in [1]. Some authors use the term *binoid*.

[2] Plonka [15] introduced the term *quasi-lattice* for these structures.

3 Labeled graphs

Suppose that A is a set. An A-*labeled graph* is a finite nonempty graph $(G, \sim_G)$ equipped with a labeling function $\ell_G : G \to A$. Here, $\sim_G \subseteq G \times G$ is an irreflexive symmetric relation on the set G of vertices. The elements of $\sim_G$ are called edges. A morphism of A-labeled graphs is a function which preserves the edges and the labeling. An isomorphism is a bijective morphism whose inverse is also a morphism. *We identify any two isomorphic A-labeled graphs.*

Suppose that G and H are A-labeled graphs. Since we work with isomorphism classes of labeled graphs, in the definitions of the $\oplus$ and $\otimes$ operations below we may without loss of generality assume that G and H are disjoint. We define

$$G \oplus H \quad = \quad (G \cup H, \sim_{G\oplus H}, \ell_{G\oplus H}),$$

where

$$\begin{aligned} \sim_{G\oplus H} &= \sim_G \cup \sim_H \\ \ell_{G\oplus H}(u) &= \begin{cases} \ell_G(u) & \text{if } u \in G \\ \ell_H(u) & \text{if } u \in H. \end{cases} \end{aligned}$$

Moreover, we define

$$G \otimes H \quad = \quad (G \cup H, \sim_{G\otimes H}, \ell_{G\otimes H}),$$

where

$$\sim_{G\otimes H} \quad = \quad \sim_G \cup \sim_H \cup G \times H \cup H \times G,$$

and where $\ell_{G\otimes H}$ is defined in the same way as $\ell_{G\oplus H}$. We also define an operation of quasi-complementation:

$$\overline{G} \quad = \quad (G, \sim_{\overline{G}}, \ell_{\overline{G}}),$$

where $\ell_{\overline{G}} = \ell_G$ and

$$\sim_{\overline{G}} \quad = \quad \{(u,v) \in G \times G : u \neq v,\ (u,v) \notin \sim_G\}.$$

The collection of all A-labeled graphs, equipped with the above operations, satisfies all of the defining equations of commutative De Morgan bisemigroups not involving 0 and 1. Thus, if we add elements 0 and 1 such that (1) – (4), (8) and (9) hold, then we obtain a commutative De Morgan bisemigroup $\mathbf{G}_A$.

Remark 3.1 The graphs in the smallest subalgebra of $\mathbf{G}_A$ containing the singletons are called labeled *cographs*, or *complement reducible graphs*. Since the complement of any singleton graph is itself, it follows that a labeled graph is a cograph if and only if it can be generated from the singletons by any two of the operations $\oplus, \otimes$ and $^-$. It is also known, see [5, 16], that a (labeled) graph is a cograph if and only if it is P_4-free, i.e., when it contains no subgraph isomorphic to a path on 4 vertices.

Remark 3.2 An A-labeled graph may also be represented as a system $(G, \approx_G, \sim_G, \ell_G)$ where G is a finite nonempty set, $\approx_G$ and $\sim_G$ are disjoint irreflexive and symmetric relations on G, and ℓ_G is a labeling function $G \to A$. Moreover, it is required that for any two distinct vertices u, v, either $u \approx v$ or $u \sim v$ holds, i.e., that $(G, \approx_G \cup \sim_G)$ is a *complete graph*. The $\oplus$ and $\otimes$ operations can then be defined so that

$$G \oplus H \quad = \quad (G \cup H, \approx_{G\oplus H}, \sim_{G\oplus H}, \ell_{G\oplus H}),$$

where

$$\begin{aligned}\approx_{G\oplus H} &= \approx_G \cup \approx_H \cup G\times H \cup H\times G\\ \sim_{G\oplus H} &= \sim_G \cup \sim_H,\end{aligned}$$

and

$$G\otimes H = (G\cup H, \sim_{G\otimes H}, \approx_{G\otimes H}, \ell_{G\otimes H}),$$

where

$$\begin{aligned}\approx_{G\otimes H} &= \approx_G \cup \approx_H\\ \sim_{G\otimes H} &= \sim_G \cup \sim_H \cup G\times H \cup H\times G,\end{aligned}$$

and where $\ell_{G\oplus H}$ and $\ell_{G\otimes H}$ are defined above. Quasi-complementation is given by

$$\overline{G} = (G, \approx_{\overline{G}}, \sim_{\overline{G}}, \ell_{\overline{G}}),$$

where $\ell_{\overline{G}} = \ell_G$ and

$$\begin{aligned}\approx_{\overline{G}} &= \sim_G\\ \sim_{\overline{G}} &= \approx_G .\end{aligned}$$

For later use we note:

LEMMA 3.3 *The following cancellation laws hold in $\mathbf{G}_A$.*

1. *If $G_1 \oplus H = G_2 \oplus H$, then $G_1 = G_2$.*
2. *If $G_1 \otimes H = G_2 \otimes H$, then $G_1 = G_2$.*

Proof. If $G_1 \oplus H$ and $G_2 \oplus H$ are isomorphic, then G_1 and G_2 have the same number of connected components. Moreover, there is a bijection between the connected components of $G_1 \oplus H$ and $G_2 \oplus H$ which assigns to any component of $G_1 \oplus H$ an isomorphic component of $G_2 \oplus H$. But then there is a similar bijection between the components of G_1 and G_2, proving that G_1 and G_2 are isomorphic. The second claim follows from the first by taking complements. $\square$

In order to represent the free commutative De Morgan bisemigroup by cographs, we modify the operation of quasi-complementation. Suppose that A is a set and $\overline{A} = \{\overline{a} : a \in A\}$ is a disjoint copy of A. We define a new quasi-complementation operation on the set of $A\cup\overline{A}$-labeled graphs. Given $G = (G, \sim_G, \ell_G)$, define $\overline{G} = (G, \sim_{\overline{G}}, \ell_{\overline{G}})$, where

$$\sim_{\overline{G}} = \{(u,v)\in G\times G : u\neq v,\ (u,v)\notin \sim_G\} \tag{11}$$

$$\ell_{\overline{G}}(u) = \overline{\ell_G(u)}, \quad u\in G. \tag{12}$$

Here we write $\overline{\overline{a}} = a$, for all $a\in A$. As before, we define $\overline{0} = 1$ and $\overline{1} = 0$. The resulting algebra, denoted $\mathbf{G}_{A,\overline{A}}$, is a again a commutative De Morgan bisemigroup. Let $\mathbf{CDBS}_A$ denote the least subalgebra of $\mathbf{G}_{A,\overline{A}}$ containing the singleton graph labeled a, for each $a\in A$. By Remark 3.1, an $A\cup\overline{A}$-labeled graph G belongs to $\mathbf{CDBS}_A$ if and only if G is P_4-free. Since $\mathbf{G}_{A,\overline{A}}$ is a commutative De Morgan bisemigroup, so is $\mathbf{CDBS}_A$. In the next result, we identify each letter in $A\cup\overline{A}$ with the corresponding labeled graph having a single vertex.

THEOREM 3.4 $\mathbf{CDBS}_A$ *is freely generated by A in the variety of all commutative De Morgan bisemigroups.*

Proof. Suppose that S is a commutative De Morgan bisemigroup and h is a function $A \to S$. We show how to extend h to a homomorphism $h^\sharp : \mathbf{CDBS}_A \to S$. First, we define $h^\sharp(0) = 0$ and $h^\sharp(1) = 1$. Moreover, we define $h^\sharp(a) = h(a)$ and $h^\sharp(\overline{a}) = \overline{h(a)}$, for each $a \in A$. Suppose now that $G \in \mathbf{CDBS}_A$ has 2 or more vertices. If G is not connected, write G in the form $G = G_1 \oplus \ldots \oplus G_n$, where the G_i are all of the connected components of G. We have $G_i \in \mathbf{CDBS}_A$, for all $i = 1, \ldots, n$. Define $h^\sharp(G) = h^\sharp(G_1) \oplus \ldots \oplus h^\sharp(G_n)$. If G is connected, then G can be written as $G = G_1 \otimes \ldots \otimes G_n$, where the G_i are disconnected. In this case, define $h^\sharp(G) = h^\sharp(G_1) \otimes \ldots \otimes h^\sharp(G_n)$. That $h^\sharp$ is well-defined follows by the associativity and commutativity of the operations. It is now immediate that $h^\sharp$ preserves $\oplus$ and $\otimes$. The fact that $h^\sharp$ also preserves quasi-complementation follows from Lemma 3.5. $\square$

LEMMA 3.5 *Suppose that S and S' are De Morgan bisemigroups. Suppose that $X \subseteq S$ is closed for quasi-complementation.*

1. *Then X generates S if and only if every $s \in S - \{0, 1\}$ can be generated from X by $\oplus$ and $\otimes$.*
2. *Suppose that S is generated by X. Then a function $h : S \to S'$ is a De Morgan bisemigroup homomorphism if and only if h preserves $0, 1$, the quasi-complementation on the elements of X, and the $\oplus$ and $\otimes$ operations.*

When A has a single element, $\mathbf{G}_A$ may be considered to be a commutative De Morgan bisemigroup of *unlabeled* graphs, and the constants 0 and 1. Let $\mathbf{G}$ denote this commutative De Morgan bisemigroup and let $\mathbf{CG}$ denote the subalgebra of $\mathbf{G}$ determined by the cographs and the constants. It is natural to ask whether there are equations that hold in $\mathbf{G}$ but fail to hold in all commutative De Morgan bisemigroups, i.e., whether the variety of commutative bisemigroups is generated by $\mathbf{G}$. Below we answer this question. We will show that both $\mathbf{G}$ and $\mathbf{CG}$ generate the variety of commutative De Morgan bisemigroups.

PROPOSITION 3.6 *When A is a countable set, there is an embedding of the free commutative De Morgan bisemigroup $\mathbf{CDBS}_A$ into $\mathbf{CG}$.*

Proof. Let $a_1, a_2, \ldots$ be a fixed enumeration of A, and for each $n \geq 1$, define

$$G_n = \bullet \otimes (\overbrace{\bullet \oplus \ldots \oplus \bullet}^{(n+1)-\text{times}}),$$

where $\bullet$ denotes the singleton graph. Note that the complement of G_n in $\mathbf{CG}$ is

$$\overline{G_n} = \bullet \oplus (\overbrace{\bullet \otimes \ldots \otimes \bullet}^{(n+1)-\text{times}}).$$

Let f denote the homomorphism $\mathbf{CDBS}_A \to \mathbf{CG}$ determined by the assignment $a_n \mapsto G_n$, $n \geq 1$.

For a graph $G \in \mathbf{CDBS}_A$, $f(G)$ can be constructed by replacing each vertex of G labeled a_n by a copy of G_n, and each vertex labeled $\overline{a_n}$ by a copy of $\overline{G_n}$. Thus, if u and v are connected by an edge in G, then any vertex of the graph replacing u will be connected in $f(G)$ to each vertex of the graph replacing v. Clearly, each graph in the image of f contains both vertices that are connected by an edge and disconnected vertices. Using this fact, it follows that when $G \in \mathbf{CDBS}_A$ is connected and G is not a singleton, then $f(G)$ contains both a complete graph on three vertices and a graph isomorphic to P_3, the path on three vertices. Also, if G is connected, then so is $f(G)$, unless G consists of a single vertex labeled $\overline{a_n}$, for some n. Thus, when G is connected and is not a singleton, no strongly connected component of $f(G)$ is isomorphic to any of the G_n or to a connected component of any of the $\overline{G_n}$.

We claim that f is injective. To prove this, suppose that $H, K \in \mathbf{CDBS}_A$, $H \neq K$ have minimal number of vertices with $f(H) = f(K)$. If H or K is a singleton, then $H = K$, since there is no nontrivial way of generating any of the G_n and $\overline{G_n}$ form the graphs $G_m, \overline{G_m}$, $m \geq 1$ by the operations $\oplus$ and $\otimes$. Thus we may assume that neither H nor K is a singleton. Suppose that H is disconnected, say $H = H_1 \oplus \ldots \oplus H_m$, where $m > 1$ and the H_i are connected. Then K is also disconnected, since otherwise $f(K)$ would be connected, but $f(H)$ is not. Let $K = K_1 \oplus \ldots \oplus K_n$, where the K_j are connected. If one of the H_i has a single vertex, then by the preceding argument, there is a j with $H_i = K_j$. Removing these components from H and K, the resulting graphs H' and K' are distinct and satisfy $f(H') = f(K')$, by Lemma 3.3. Since also $H' \neq K'$, this contradicts our assumption on H and K. We conclude that none of the H_i is a singleton. In the same way, none of the K_j is a singleton. But then all of the graphs $f(H_i)$ and $f(K_j)$ are connected, so that $f(H) = f(K)$ only if $m = n$ and there is a permutation $i_1, \ldots, i_n$ of the integers $1, \ldots, n$ such that $f(H_j) = f(K_{i_j})$, for all $j = 1, \ldots, n$. But since $H \neq K$, there is a j with $H_j \neq K_{i_j}$. This is again a contradiction. If H is connected, consider $\overline{H}$ and $\overline{K}$. They have the same number of vertices as H and K, and $f(\overline{H}) = f(\overline{K})$. But since $\overline{H}$ is disconnected, we can derive a contradiction as before. $\square$

THEOREM 3.7 *The variety of commutative De Morgan bisemigroups is generated by either one of the algebras* $\mathbf{G}$ *and* $\mathbf{CG}$.

Proof. Since $\mathbf{CG}$ is a subalgebra of $\mathbf{G}$ and $\mathbf{G}$ is a commutative De Morgan bisemigroup, it suffices to prove that the variety generated by $\mathbf{G}$ contains the free commutative De Morgan bisemigroup $\mathbf{CDBS}_A$ generated by a countable set A. But this holds by Proposition 3.6. $\square$

4 Labeled directed graphs

In order to give a representation of the free De Morgan bisemigroups, we will now consider *labeled 2-digraphs* $(G, \rho_G, \tau_G, \ell_G)$, where G is a finite nonempty set, ρ_G and τ_G are irreflexive antisymmetric relations on G, and $\ell_G : G \to A$. We also require that for any two distinct vertices $u, v \in G$, either u and v are related by ρ_G, or else u and v are related by τ_G, but not by both. An isomorphism $G \to H$ of A-labeled digraphs G, H is a bijection which preserves the edges and the labeling, i.e., a bijective function $f : G \to H$ such that for all $u, v \in G$, $(u, v) \in \rho_G$ if and only if $(f(u), f(v)) \in \rho_H$ and $\ell_G(u) = \ell_H(f(u))$. It then follows that $(u, v) \in \tau_G$ if and only if $(f(u), f(v)) \in \tau_H$. We identify any two isomorphic A-labeled 2-digraphs. The $\oplus$ and $\otimes$ operations are defined as follows, where without loss of generality we again assume that G and H are disjoint.

$$G \oplus H = (G \cup H, \rho_{G\oplus H}, \tau_{G\oplus H}, \ell_{G\oplus H}),$$

where

$$\begin{aligned} \rho_{G\oplus H} &= \rho_G \cup \rho_H \cup G \times H \\ \tau_{G\oplus H} &= \tau_G \cup \tau_H, \end{aligned}$$

and

$$G \otimes H = (G \cup H, \rho_{G\otimes H}, \tau_{G\otimes H}, \ell_{G\otimes H}),$$

where

$$\begin{aligned} \rho_{G\otimes H} &= \rho_G \cup \rho_H \\ \tau_{G\otimes H} &= \tau_G \cup \tau_H \cup G \times H, \end{aligned}$$

and where $\ell_{G\oplus H}$ and $\ell_{G\otimes H}$ are defined above. Quasi-complementation is given by

$$\overline{G} \;=\; (G, \rho_{\overline{G}}, \tau_{\overline{G}}, \ell_{\overline{G}}),$$

where $\ell_{\overline{G}} = \ell_G$ and

$$\begin{aligned} \rho_{\overline{G}} &= \tau_G \\ \tau_{\overline{G}} &= \rho_G. \end{aligned}$$

Let $\mathbf{D}_A$ denote the structure that results by adding 0 and 1 to A-labeled 2-digraphs such that (1) – (4) and (8), (9) hold. Clearly, $\mathbf{D}_A$ is a De Morgan bisemigroup. When $\overline{A}$ is a disjoint copy of A, we may also define the De Morgan bisemigroup $\mathbf{D}_{A,\overline{A}}$ which is the same as $\mathbf{D}_{A\cup\overline{A}}$ except that the labeling function of the quasi-complement is given by (12). Let $\mathbf{DBS}_A$ denote the subalgebra of $\mathbf{D}_{A,\overline{A}}$ generated by the singleton 2-digraphs corresponding to the elements of A.

Remark 4.1 The paper [6] contains a common generalization of the geometric characterization of series-parallel digraphs (or posets) [9, 16] and cographs [5, 16]. It follows from this general result that an A-labeled 2-digraph $(G, \rho_G, \tau_G, \ell_G)$ belongs to the subalgebra of $\mathbf{D}_A$ generated by the singletons if and only if both ρ_G and τ_G are transitive (so that they define partial orders), and ρ_G is N-free. Thus there are no distinct vertices u, v, w, z such that the order relations between them are given by $u\rho_G w$, $v\rho_G w$, $v\rho_G z$. It then follows τ_G is also N-free. The same conditions characterize the 2-digraphs in $\mathbf{DBS}_A$.

Theorem 4.2 $\mathbf{DBS}_A$ *is freely generated by A in the class of all De Morgan bisemigroups.*

The proof is similar to that of Theorem 3.4.

Consider now the De Morgan bisemigroup $\mathbf{D}$ of unlabeled 2-digraphs, and its subalgebra $\mathbf{ND}$ generated by the singleton 2-digraph. We have:

Theorem 4.3 *The variety of De Morgan bisemigroups is generated by both* $\mathbf{D}$ *and* $\mathbf{ND}$.

This can be proven following the lines of the proof of Proposition 3.6 and Theorem 3.7. One shows that when A is a countable set $\{a_1, a_2, \ldots\}$, the homomorphism $g : \mathbf{DBS}_A \to \mathbf{ND}$ determined by the assignment

$$a_n \;\mapsto\; \bullet \oplus (\overbrace{\bullet \otimes \ldots \otimes \bullet}^{n-\text{times}}, \quad n \geq 1)$$

is injective. Indeed, this fact follows from Proposition 3.6, since the homomorphism f given in the proof of Proposition 3.6 is the composite of the homomorphism g with a homomorphism $\mathbf{CG} \to \mathbf{ND}$.

5 Free De Morgan bisemilattices

In order to obtain a geometric representation of the free De Morgan bisemilattices, we will consider labeled graphs with a particular property. We call an A-labeled graph G *$\oplus$-irreducible* if G is connected, i.e., when there exist no labeled graphs G_1 and G_2 with $G = G_1 \oplus G_2$. Similarly, we call G $\otimes$-irreducible, if $\overline{G}$ is $\oplus$-irreducible, i.e., when there exist no labeled graphs G_1 and G_2 with $G = G_1 \otimes G_2$. If G is both $\oplus$-irreducible and $\otimes$-irreducible, then we call G *irreducible.* The $\oplus$-components of G are the connected components of G. The $\otimes$-components of G are the quasi-complements of the $\oplus$-components of $\overline{G}$. Thus, denoting the $\oplus$-components by G_i and the $\otimes$-components by H_j, where $i = 1, \ldots, n$ and $j = 1, \ldots, m$, we can write

$$\begin{aligned} G &= G_1 \oplus \ldots \oplus G_n \\ G &= H_1 \otimes \ldots \otimes H_m, \end{aligned}$$

where each G_i is $\oplus$-irreducible and each H_j is $\otimes$-irreducible.

We say that an A-labeled graph G has *inherently nonisomorphic components* if it is irreducible, or for each way of writing $G = G_1 \oplus \ldots \oplus G_n$ or $G = G_1 \otimes \ldots \otimes G_n$, the labeled graphs G_i are pairwise nonisomorphic and have inherently nonisomorphic components. Clearly, if G is not $\oplus$-irreducible then G has inherently nonisomorphic components if and only if the $\oplus$-components of G are pairwise nonisomorphic and have inherently nonisomorphic components. And if G is not $\otimes$-irreducible then G has inherently nonisomorphic components if and only if the $\otimes$-components of G are pairwise nonisomorphic and have inherently nonisomorphic components.

Suppose that G, H have inherently nonisomorphic components. Up to isomorphism, let $C_1, \ldots, C_m$ denote all of the $\oplus$-components of G and H, and $D_1, \ldots, D_n$ the $\otimes$-components of G and H. We define

$$\begin{aligned} G \sqcup H &= C_1 \oplus \ldots \oplus C_m \\ G \wedge H &= D_1 \otimes \ldots \otimes D_n. \end{aligned}$$

It is easy to see that G has inherently nonisomorphic components if and only if $\overline{G}$ has. It follows that the quasi-complementation operation is well-defined on A-labeled graphs having inherently nonisomorphic components. When we add 0 and 1, there results a De Morgan bisemilattice $\mathbf{IG}_A$. In a similar way, we can define the De Morgan bisemilattice $\mathbf{IG}_{A,\overline{A}}$.

For any $G, H \in \mathbf{IG}_A$ or $G, H \in \mathbf{IG}_{A,\overline{A}}$,we have $G \sqsupseteq H$ if and only if $G = 1$ or $H = 0$ or every $\oplus$-component of H is a $\oplus$-component of G, and $G \leq H$ if and only if $G = 0$ or $H = 1$ or every $\otimes$-component of H is a $\otimes$-component of G. (The relations $\sqsupseteq$ and $\leq$ were defined in Section 2.) Thus, $G \sqcap H = \inf_{\sqsupseteq}\{G, H\}$ always exists. Moreover, when $G \sqcap H$ is a graph, its set of $\oplus$-components is the intersection of the sets of $\oplus$-components of G and H. In the same way, $G \vee H = \sup_{\leq}\{G, H\}$ also exists.

Proposition 5.1 *Suppose that $G, H \in \mathbf{IG}_A$, or $G, H \in \mathbf{IG}_{A,\overline{A}}$. If G and H are cographs, then $G \sqcup H$ and $G \wedge H$ are also cographs. Moreover, $G \sqcap H$ and $G \vee H$ are either cographs or belong to $\{0, 1\}$.*

Proof. $G \sqcup H$ contains a P_4 if and only if G or H does, and similarly for $G \wedge H$. Moreover, $G \sqcap H$ contains a P_4 if and only if both G and H do, and similarly for $G \vee H$. □

Thus, those $A \cup \overline{A}$-labeled cographs that have inherently nonisomorphic components form a subalgebra of $\mathbf{IG}_{A,\overline{A}}$. Let $\mathbf{DBSL}_A$ denote this subalgebra.

Proposition 5.2 *For any $G \in \mathbf{G}_{A\overline{A}}$, $G \in \mathbf{DBSL}_A$ if and only if G can be generated from the singletons corresponding to the letters in A and the constants 0, 1 by the operations $\sqcup, \wedge$ and $^-$.*

Thus, $\mathbf{DBSL}_A$ is the subalgebra of $\mathbf{IG}_{A,\overline{A}}$ generated by the singletons corresponding to the letters in A.

Theorem 5.3 *For each set A, $\mathbf{DBSL}_A$ is the free De Morgan bisemilattice on A.*

Proof. We have already noted that $\mathbf{DBSL}_A$ is a De Morgan bisemilattice. By Theorem 3.4, there is a homomorphism $h : \mathbf{CDBS}_A \to \mathbf{DBSL}_A$ which maps the labeled graph corresponding to each letter in A to itself. By Proposition 5.2, h is surjective. To complete the proof we need to show that whenever θ is a congruence relation on $\mathbf{CDBS}_A$ such that the quotient $\mathbf{CDBS}_A/\theta$ is a De Morgan bisemilattice, i.e., such that $\oplus$ and $\otimes$ are idempotent on $\mathbf{CDBS}_A/\theta$, then the kernel of h is included in θ. But this follows from the fact that for all such congruence relations θ and any $G \in \mathbf{CDBS}_A$, $G\,\theta\, h(G)$. Indeed, this is clear when G is a singleton. We proceed by induction on the number of vertices of G. When G has two or more vertices, then G is either not $\oplus$-irreducible or not $\otimes$-irreducible. We only consider the first case. If G is not $\oplus$-irreducible, then we can write $G = G_1 \oplus \ldots \oplus G_n$, $n > 1$, where the G_i are $\oplus$-irreducible labeled graphs in $\mathbf{CDBS}_A$. By induction, $G_i\,\theta\, h(G_i)$ holds for each $i = 1, \ldots, n$. Let $\{i_1, \ldots, i_m\}$ denote a maximal subset of $\{1, \ldots, n\}$ such that $h(G_{i_1}), \ldots, h(G_{i_m})$ are pairwise nonisomorphic. Since the $h(G_i)$ are connected, we have

$$\begin{array}{rcl} G & = & G_1 \oplus \ldots \oplus G_n \\ & \theta & h(G_1) \oplus \ldots \oplus h(G_n) \\ & \theta & h(G_{i_1}) \oplus \ldots \oplus h(G_{i_m}) \\ & = & h(G_1) \sqcup \ldots \sqcup h(G_n) \\ & = & h(G). \quad \square \end{array}$$

Call a De Morgan bisemilattice S a *De Morgan bilattice* [2] if $x \sqcap y$ and $x \vee y$ exist for all $x, y \in S$. Moreover, call a De Morgan bilattice S *locally distributive* [2] if $(S, \sqcup, \sqcap)$ and $(S, \wedge, \vee)$ are distributive lattices.

COROLLARY 5.4 *Every free De Morgan bisemilattice is a locally distributive De Morgan bilattice.*

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